

Geometry of Spin and Spin^c structures in the M-theory partition function

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Abstract

We study the effects of having multiple Spin structures on the partition function of the spacetime fields in M-theory. This leads to a potential anomaly which appears in the eta-invariants upon variation of the Spin structure. The main sources of such spaces are manifolds with nontrivial fundamental group, which are also important in realistic models. We extend the discussion to the Spin^c case and find the phase of the partition function, and revisit the quantization condition for the C -field in this case. In type IIA string theory in ten dimensions, the mod 2 index of the Dirac operator is the obstruction to having a well-defined partition function. We geometrically characterize manifolds with and without such an anomaly and extend to the case of nontrivial fundamental group. The lift to KO-theory gives the α -invariant, which in general depends on the Spin structure. This reveals many interesting connections to positive scalar curvature manifolds and constructions related to the Gromov-Lawson-Rosenberg conjecture. In the twelve-dimensional theory bounding M-theory, we study similar geometric questions, including choices of metrics and obtaining elements of K-theory in ten dimensions by pushforward in K-theory on the disk fiber. We interpret the latter in terms of the families index theorem for Dirac operators on the M-theory circle and disk. This involves superconnections, eta-forms, and infinite-dimensional bundles, and gives elements in Deligne cohomology in lower dimensions. We illustrate our discussion with many examples throughout.

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1 Introduction and Summary

Index theorems play an important role in characterizing anomalies and zero modes in theories in physics. The Atiyah-Singer (AS) index theorem in even dimensions, in the absence of boundaries, involves invariants which do not depend on the choice of geometric notions such as a connection or a Spin structure. However, the situation in the presence of a boundary is drastically different, as the contribution from the boundary depends on geometric and analytical quantities in a precise way. M-theory is described by this setting [165] of the Atiyah-Patodi-Singer (APS) index theorem [11] [12] [13].

The definition and dynamics of M-theory remain a mystery. While there is mounting evidence that this theory exists, there is yet no intrinsic understanding of this theory that does not use the ‘corners of the moduli space’ such as ten-dimensional superstring theories or eleven-dimensional supergravity. A semi-classical approach to M-theory involves ‘continuing’ the eleven-dimensional supergravity action to the quantum regime. Of interest is the topological part I_{top} , comprised of the Chern-Simons term [42] and the one-loop term [159] [61]. I_{top} was shown by Witten [165] to be written as a sum of the index of a Dirac operator coupled to an E_8 bundle and a Rarita-Schwinger operator. Part of the motivation was first the fact that the cohomology class of the degree four field G_4 in M-theory contains, as a summand, a characteristic class of an E_8 bundle, and second, existence of the $E_8 \times E_8$ heterotic string theory on the boundary [96].

The dimensional reduction of this E_8 gauge theory from the total space Y^{11} of a circle bundle to ten dimensions was performed by [54] where the partition function was matched (in a limit) with the corresponding K-theoretic partition function of the Ramond-Ramond fields in type IIA string theory. Many subtleties are involved in this quantum equivalence. The partition function of the C -field C_3 and the Rarita-Schwinger field ψ_1 can be factorized into a modulus and a phase, the latter being given by [54]

$$\Phi = \exp \left[2\pi i \left(\frac{1}{2} \bar{\eta}_{E_8} + \frac{1}{4} \bar{\eta}_{RS} \right) \right] = \exp \left[2\pi i \left(\frac{1}{4} (h_{E_8} + \eta_{E_8}) + \frac{1}{8} (h_{RS} + \eta_{RS}) \right) \right] , \quad (1.1)$$

where $\bar{\eta}_V = \frac{1}{2}(\eta_V + \dim \text{Ker} D_V) = \frac{1}{2}(\eta_V + h_V)$ is the reduced eta invariant of the Dirac operator D_V coupled to the bundle V , with V being the E_8 bundle or the Rarita-Schwinger bundle (i.e. $\text{RS} = TY^{11} - 3\mathcal{O}$, with \mathcal{O} a trivial real line bundle).

Taking $Y^{11} = X^{10} \times S^1$, the C -field, being a pullback from X^{10} , has an orientation-reversing symmetry. Under this symmetry, the Chern-Simons term reverses sign, and the phase of the partition function is complex-conjugated. The Dirac operator changes sign under reflection of one coordinate, so the nonzero eigenvalues appear in pairs $(\lambda, -\lambda)$, so that the eta invariants are zero [54]. The M-theory/type IIA connection via E_8 gauge theory was further studied in [116] and incorporated into the description of the twisted K-theory partition function in the presence of the Neveu-Schwarz (NS) H-field H_3 (or its ‘potential’, the B -field), as well as in [148] in relation to gerbes. In this paper we explore effects related to Spin and Spin^c structures.

In the above works both the ten- and eleven-dimensional manifolds are taken to be Spin. In this case we study the dependence of the phase of the partition function on the choice of Spin structure (starting in section 2). We then work out an extension to the Spin^c case (starting in section 2.6 and then more extensively in section 3). We study how Spin and Spin^c structures in eleven dimensions and ten dimensions are related,

especially in the context of the partition function. First, we look at the case when the ten-dimensional manifold is Spin^c and then when both the ten- and eleven-dimensional manifolds are Spin^c .

Outline. This paper is composed of two intermixed parts: exposition and original results. The first part is written with an emphasis on a contextual point of view (and hence is not traditional) and involves:

- Recasting physical results in a mathematical (and invariant) language, thus highlighting otherwise hidden structures.
- Bringing into physics results from mathematics to give constructive descriptions of structures appearing from the physics side.

We hope this makes the paper more self-contained. Thus, this paper can be used by physicists interested in seeing how geometric ideas can be used in an effective way in constructing and studying the partition function as well as by mathematicians who are interested in applications and examples. The second part on original research, which is the larger part of the paper, has as main points the following

1. *Highlight the importance of Spin structures in M-theory.* When multiple Spin structures exist, the different Spin structures may lead to different answers for analytical and geometric entities, and hence for physical entities (section 2). For example, the mod 2 index of the Dirac operator in dimensionally-reduced M-theory to ten dimensions admits a lift to KO-theory whose value depends on the choice of Spin structure (section 5). This suggests that when calculating the *full* partition function, which includes moduli of the metric, one should also take contributions from the corresponding different Spin structures into account in the sum. Furthermore, considering the bounding (as opposed to the nonbounding) Spin structure on the M-theory circle leads to many interesting connections to global analysis: eta-forms, adiabatic limits, and superconnections (section 7).
2. *Study the partition function on manifolds which are not simply connected.* Manifolds with multiple Spin structures are non-simply connected. The main examples are spherical space forms S^n/Γ , which in even dimensions can only be projective spaces. Representations of the finite group Γ provide possibly multiple Spin structures. At the level of bundles this can be encoded by flat vector bundles and their holonomy. We identify the fields in ten and eleven dimensions which could support nontrivial fundamental group. This is the subject of section 2 and then sections 6.1 and 6.2. Framed manifolds, considered at the end of section 6.2, provide interesting examples. We also do the same for the second homotopy groups, which is relevant for Spin^c structures, in section 3 and section 6.3.
3. *Study the relation between Spin and Spin^c structures on one hand on Y^{11} and the corresponding structures on X^{10} on the other hand.* We consider type IIA string theory as obtained from M-theory via quotient by a circle action. The spaces are Spin or Spin^c depending on whether the circle action lifts to the Spin bundle. We start with an eleven-dimensional Spin manifold. For an action of even type, or projectable action, the ten-dimensional quotient is Spin. For an action of odd type, or nonprojectable action, the ten-dimensional quotient is Spin^c . We study both cases in detail in section 2 (especially sections 2.5 and 2.6) and section 3. Important examples in eleven dimensions include contact manifolds and spherical space forms. Important examples in ten dimensions include Kähler manifolds.
4. *Study the dependence of the phase of the partition function on the Spin structure.* The phase of the partition function in M-theory is given in terms of exponentiated eta invariants in equation (1.1). These are geometric/analytical as opposed to topological invariants. We ask whether this depends on the choice of Spin structure. We perform the analysis for both operators. In order to illustrate the point, we work out examples in eleven and seven dimensions showing explicitly how the eta function depends on the choice of Spin structure. We also extract the number of zero modes of the corresponding Dirac operators, as these appear in the phase in section 2.7.

5. *Extend the description of the phase of the partition function to the Spin^c case.* We do this in section 3. We motivate the case for Spin^c in ten, eleven, and twelve dimensions in section 2.6 and section 3.2. Important examples in odd dimensions are contact manifolds and spherical space forms, the motivation for both of which we highlight in section 3.3. In even dimensions, prominent examples are Kähler manifolds and coboundaries of spherical space forms. We also study situations of dependence on almost complex structures, which are closely related to Spin^c structures (section 4.1). This makes connection to obstructions and generalized cohomology in section 3.5. The phase in the adiabatic limit, i.e. when the volume of the base ten-dimensional manifold is taken to be large with respect to the volume of the (circle) fiber, turns out to admit an expression which is much more involved than that of the Spin case. We do this in section 3.8, building on discussions from section 3.6. This leads to nontrivial congruences that depend on the M-theory circle bundle, the line bundle associated to the Spin^c -structure, the degree four characteristic class a of the E_8 bundle E , and the Pontrjagin classes of the ten-dimensional manifold X^{10} . We illustrate how the M-theory circle can be used to define the Spin^c structure. We also consider the partition functions of the M2-brane and the M5-brane in the Spin^c case in section 3.7.

The main feature of our discussion is that we write the phase of the eleven-dimensional expression as an integral of quantities in ten dimensions. Thus that makes a departure from the discussion of [54] in two respects: We are looking at the phase in eleven dimensions, involving the eta invariant, instead of the phase in twelve dimensions, involving the index, and the expression is a ten-dimensional integral. This makes connection to the treatment in [116] using eta-forms.

6. *Highlight the importance of geometry, especially Ricci and scalar curvatures, in the eta invariant and the mod 2 index appearing in the construction of the partition function.* We do this starting in section 5.1. The mod 2 index appears in the K-theoretic anomaly of the Ramond-Ramond fields in ten dimensions [54]. In section 5.2 we characterize the mod 2 index in ten dimensions from topological, analytical, and geometric angles. The mod 2 index in dimensions 2 and 10 can be viewed as an instance of the \hat{A} -genus and admits a lift to KO-theory called the α -invariant, which depends on the choice of Spin structure on the ten-dimensional manifold. Requiring this invariant to vanish is relating to positive scalar curvature (psc) metrics via the Gromov-Lawson-Rosenberg conjecture. We use the notion of a Clifford-linear Dirac operator [107] which, in a sense, unifies all Spin representations. We give many examples that are important in type IIA string theory (starting earlier in section 4.4). The kernel of the α invariant, which is the lift of the \hat{A} -genus, determines the situations for which the mod 2 index of the Dirac operator vanishes, and hence for which the partition function in ten dimensions is anomaly-free. Total spaces of $\mathbb{H}P^2$ bundles give representatives in each cobordism class of $\ker \alpha$. We do this also for the non-simply connected case. The discussion (in section 5.2) then naturally relates structures in ten dimensions to structures in two dimensions. As far as the geometry of the bounding twelve-manifold is involved, we investigate how that is related to the physical problem in the context of the Atiyah-Patodi-Singer index theorem, and find in section 5.3 that there are preferred metrics that take into account the fibration structure between type IIA string theory and M-theory.
7. *Extend the constructions to the family case.* We model the physical setting in terms of families of Dirac operators on the M-theory circle (i.e. the eleventh direction) parametrized by the ten-dimensional base, home to type IIA string theory. This leads to several interesting consequences:
 - In section 7.2 we explain physically the role of the Quillen and Bismut superconnections. Because of nonchirality, type IIA is a natural setting for the superconnection formalism. The superconnection represents the geometric description of the pushforward map in K-theory. Thus, starting from bundles on the fiber in the M-theory compactification, we can produce K-theory elements in the base. They lead to a sort of family version of the dilatino supersymmetry transformation. We also characterize the corresponding infinite-dimensional bundles.
 - In section 7.2.2 we consider a generalization of usual dimensional reduction mechanisms. The Scherk-Schwarz dimensional reduction uses an abelian rigid symmetry of the equations of motion to

generalize the reduction ansatz by allowing a *linear* dependence on the coordinate of the fiber. For example $v(x, z) = mz + v(x)$. What we consider is a generalization of this so that the dependence on the fiber coordinate is not necessarily linear, and also allows the base spacetime derivatives of the fiber vector $\nabla_\mu v_z$ to be nonzero in general. The generalization of the second type has been considered in [6] where it was related to metric torsion. The general case we consider will also involve torsion in an intimate way, which will essentially be the RR 2-form F_2 .

- The adiabatic limit (physically in our interpretation: the semiclassical limit) of the eta invariant is given in terms of the Bismut-Cheeger eta-forms. We characterize these physically and provide some examples in section 7.3.

- The eta-forms lead to gerbes and Deligne cohomology classes on the base. Varying the dimension of the fiber (mostly either one- or two-dimensional) we get gerbes which are either of even or odd degree. We characterize these physically in section 7.4. We also consider the new feature of taking into account contributions from the kernels of vertical Dirac operators.

- Starting in section 7.2.2, we consider the pushforward of bundles in eleven dimensions to bundles in ten (and lower) dimensions at the level of K-theory. Several new features arise, including the appearance of infinite-dimensional bundles.

8. *Highlight consequences for the fields in ten and eleven dimensions.* We focus on the M-theory circle. From the point of view of the ten-dimensional type IIA string theory, this means that we are considering the Ramond-Ramond (RR) 1-form potential, corresponding to the connection on the circle bundle, and its curvature 2-form RR field strength F_2 . From a homological point of view, the object that carries a charge with respect to F_2 is the D0-brane [138]. Such branes carry K-theory charges [122]. On the other hand, such charges are associated with the Kaluza-Klein (Fourier) modes for the reduction on the circle. We argue in section 7.4 that generation of such modes corresponds to the Adams operations on the corresponding line bundle. This operation is an automorphism in K-theory which raises the charge of the brane by k units, which in turn corresponds to the k th Kaluza-Klein mode.

- The B -field in ten dimensions is essentially the integration of the C -field over the circle fiber. In section 7.1 we characterize this as the harmonic representative of the Spin^c structure in the adiabatic limit. This is analogous to the characterization of the C -field as (essentially) a harmonic representative of the String class in [150].

- The Ramond-Ramond fields: the K-theory class in ten dimensions corresponding to the family of Dirac operators can be obtained. This suggests that one can set up a formalism which relates M-theory and type IIA string theory in K-theory directly without having to pass through cohomology.

- The dilaton plays an important role in our discussion starting in sections 2.4, 2.5, 2.6 and then in sections 7.2 and 7.3. First, it represents the volume of the circle fiber, so that its size measures the string coupling. We use this to give the eigenvalues of the dimensionally reduced Dirac operators in the weak string coupling limit. Second, in the expression of the eta-forms, a scalar parameter appears, which we identify essentially with the dilaton. This suggests the interpretation of the integral defining the eta-form to be an integral over all possible sizes of the M-theory circle, and hence over all values of the string coupling constants (at least formally).

- The dilatino – the superpartner of the dilaton – also has an interesting description in our setting (section 7.2). Its supersymmetry transformations, which is essentially the difference between the Lie derivative with respect to the eleventh direction and the covariant derivative, admit a generalization to the families case. It becomes essentially the difference between the Bismut superconnection and the Quillen superconnection.

- The C -field in M-theory admits a shifted quantization condition in the Spin case [165]. We extend this to the Spin^c case in section 3.7. M-theory admits a parity symmetry. We consider the effect of this symmetry on Spin structures and eta forms in section 2.3 and 7.3.

- We highlight with many concrete examples throughout. We hope that such examples could be useful for future investigations, especially in relation to realistic dimensional reduction to lower dimensions.

Spin structures. A *Spin structure* on a n -dimensional manifold M consists of a $\text{Spin}(n)$ principal bundle $P_{\text{Spin}}(M)$ over M together with a two-fold covering map $\rho : \text{Spin}(n) \rightarrow \text{SO}(n)$ such that the diagram

$$\begin{array}{ccc}
 P_{\text{Spin}}(M) \times \text{Spin}(n) & \longrightarrow & P_{\text{Spin}}(M) \\
 \downarrow \varphi \times \rho & & \downarrow \varphi \\
 P_{\text{SO}}(M) \times \text{SO}(n) & \longrightarrow & P_{\text{SO}}(M)
 \end{array}
 \quad \begin{array}{c}
 \nearrow \\
 M \\
 \nwarrow
 \end{array}
 \quad (1.2)$$

commutes. The horizontal arrows are given by principal bundle structure. The set of Spin structures on a manifold M is a torsor over $H^1(M; \mathbb{Z}_2)$. Spin^c structures are complex analogs of Spin structures and are defined and studied in section 3.1. Extensive discussions on Spin and Spin^c structures can be found in [107].

Spin vs. Spin^c and ten vs. eleven dimensions. Consider the circle bundle $S^1 \rightarrow Y^{11} \rightarrow X^{10}$ with projection π . Let \mathcal{L} be the complex line bundle associated to π with first Chern class $e = c_1(\mathcal{L})$. Since $TY^{11} \cong \pi^*(TX^{10}) \oplus TS^1$, the tangent bundle along the fibers is trivial with trivialization provided by the vector field generating the S^1 -action on Y^{11} .

1. Hence a Spin structure on X^{10} – whenever it exists, i.e. when $w_2(X^{10}) = 0$ – induces a Spin structure on Y^{11} which we denote σ_1 .
2. If $w_2(X^{10}) = e \bmod 2$, i.e. X^{10} admits a Spin^c structure, then $TX^{10} \oplus \mathcal{L}$ admits a Spin structure. The choice of such a Spin structure gives a Spin structure on the disk bundle $\mathbb{D}\mathcal{L}$. Denote by σ_2 the restriction of the latter to the sphere bundle $S\mathcal{L} = Y^{11}$.
3. If $w_2(X^{10}) = 0$, i.e. if X^{10} is Spin, and if $e = 0 \bmod 2$, then σ_1 and σ_2 are *different* Spin structures on Y^{11} since the restriction of σ_1 to the fiber S^1 is the nontrivial Spin structure, which does not extend over \mathbb{D}^2 , whereas the restriction of σ_2 does by construction.

We will consider Spin structures and Spin^c structures extensively in section 2 and section 3, respectively.

Reality check 1: Multiple Spin structures and the standard model from string theory. We will see in section 2.1 that the main source of multiple Spin structures is spaces with nontrivial fundamental group. Such spaces do appear in string theory in trying to connect to four-dimensional physics. In fact, in a sense, they are even more desirable. Heterotic vacua (related to other string theories and to M-theory via dualities) on a Calabi-Yau manifold M require a nontrivial fundamental group $\pi_1(M) \neq 1$ in order to produce the standard model gauge group, as only such manifolds will allow Wilson lines [163] [153] [64] [38]. For example, torus fibered Calabi-Yau threefolds with $\pi_1(M) = \mathbb{Z}_2$ admit stable, holomorphic vector bundles with structure group $SU(5)$ [56] [57] [58]. Torus fibered Calabi-Yau threefolds with $\pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ admit stable, holomorphic vector bundles with structure group $SU(4)$ [130] [131] [55]. Examples with $\pi_1(M) = \mathbb{Z}_3 \times \mathbb{Z}_3$ are given in [37]. In all cases the groups are broken down to the standard model group.

Reality check 2: Multiple Spin structures and cosmology. There is a great deal of activity in trying to figure out the topology of the spatial part of the universe. Theoretically, isotropy and homogeneity suggest that the universe be of constant sectional curvature. Observation suggests that the universe is almost flat, but not quite, i.e. is probably curved with a very small curvature. Recent WMAP data suggests that the universe is a spherical space form [40] or a hyperbolic space form with “thorned topology” [16]. In both cases, the spaces have nontrivial fundamental groups, and this is nonabelian –the group E_8 – in the first case. There is no definite answer on whether or not the universe is simply connected, but the above results/suggestions seem reasonable. If this is the case, then the universe will admit multiple Spin structures, and the question of distinguishing them becomes of great importance.

2 M-Theory on Eleven-Dimensional Spin Manifolds

2.1 Multiple Spin structures

If a manifold is not simply connected then it may have more than one Spin structure [107] (see section 2.2). The number of Spin structures is given by $|H^1(M; \mathbb{Z}_2)|$. In particular, a simply connected manifold has at most one Spin structure.

Such a multiplicity of Spin structures occurs and is important on the string worldsheet [151]; it explains the need for the GSO projection. In the case of chiral bosons, there is a partition function for each choice of Spin structure. Different partition functions are given by theta functions, with the ones corresponding to different Spin structures permuted by $SL(2; \mathbb{Z})$ [166]. Similarly for chiral p -forms [166]. What we are considering here are Spin structures in spacetime. Such a situation can and does occur in realistic scenarios. For example (see also the end of the introduction above)

1. Every compact, locally irreducible, Ricci-flat manifold of non-generic holonomy is a Spin manifold provided that it is simply connected. In the non-simply connected case, M may have several Spin structures.
2. A Calabi-Yau manifold K with $SU(3)$ holonomy always has a Spin structure even when it is not simply connected, in which case it has several Spin structures.

The physical significance of multiple Spin structures over spacetime seems far less clear than the case of the worldsheets. In fact, in [117] a way was designed to actually avoid having multiple such structures. On the other hand, we saw in the introduction that manifolds with nontrivial fundamental group are essential for model building. One goal of this article is, to some extent, to try to fill this gap by highlighting the significance of multiple Spin structures in spacetime and providing ways of studying them in the context of partition functions in M-theory and string theory. That is, we study the consequences of allowing multiple Spin structures. We first consider several examples which will be useful later in the paper.

Spin structures on the circle. The simplest example of a manifold with more than one Spin structure is the circle, which has two inequivalent Spin structures (see [107]). The bundle of orthonormal frames is $P_{SO}(S^1) \cong S^1$. Since both S^1 and $SO(2)$ can be identified with $U(1)$, and $\text{Spin}(1) = \mathbb{Z}_2$, these structures are given by the maps

$$U(1) \times \mathbb{Z}_2 \xrightarrow{\text{pr}_1} U(1) \xrightarrow{\text{id}} U(1) \quad (2.1)$$

and

$$U(1) \xrightarrow{\text{square}} U(1) \xrightarrow{\text{id}} U(1) \quad (2.2)$$

A Spin structure on S^1 is a real line bundle L on S^1 together with an isomorphism $L \otimes L \cong TS^1$. Two choices are possible for L :

1. *Periodic*: $S^1_P = (S^1, L_P)$.
2. *Antiperiodic* (or *Möbius*): $S^1_A = (S^1, L_A)$.

The above cases can be looked at from the point of view of cobordism (see section 3.5 for more on Spin cobordism). Start with the case when the circle is a boundary of a two-disk. A frame for the disk is formed by the tangent vector to the boundary S^1 and the unit normal vector. Going around the boundary once gives a loop in the frame bundle of the disk whose lift to the Spin structure does not close up. Since $S^1 = \partial \mathbb{D}^2$ and \mathbb{D}^2 admits a unique Spin structure, S^1 as a boundary of the disk gives a two-fold connected covering of S^1 which corresponds to the trivial element in $\Omega_1^{\text{Spin}} \cong \mathbb{Z}_2$. The two-fold disconnected covering of S^1 is formed by two copies of S^1 with opposite orientations obtained by circle inversion. This covering cannot be represented as a boundary of a Spin manifold, and hence represents the generator σ of Ω_1^{Spin} . Twice

this generator, i.e. 2σ , is zero in Spin cobordism, since it bounds two cylinders, as two copies of the trivial structure on the disk bound one cylinder.

Since $S^1 \cong SO(2)$, $S^1 \cong \text{Spin}(2)$, the exact sequence $0 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(2) \rightarrow SO(2) \rightarrow 0$ takes $z \in \text{Spin}(2)$ to z^2 . For every $n \geq 1$, $N = \binom{n}{2}$, $P_{SO(N)}(SO(n)) = SO(n) \times SO(N)$ and the two coverings are

$$\begin{aligned} P_{\text{Spin}(N)}^1(SO(n)) &= SO(n) \times \text{Spin}(N) \\ P_{\text{Spin}(N)}^2(SO(n)) &= (\text{Spin}(n) \times \text{Spin}(N))/\mathbb{Z}_2, \end{aligned} \quad (2.3)$$

where \mathbb{Z}_2 acts on $\text{Spin}(n) \times \text{Spin}(N)$ by the map $(\varepsilon_1, \varepsilon_2) \mapsto (-\varepsilon_1, -\varepsilon_2)$. Thus, for $n = 2$, $N = 1$,

$$\begin{aligned} P_{\text{Spin}(1)}^1(SO(2)) &= SO(2) \times \text{Spin}(1) \\ P_{\text{Spin}(1)}^2(SO(2)) &= (\text{Spin}(2) \times \text{Spin}(1))/\mathbb{Z}_2, \end{aligned} \quad (2.4)$$

which translates into

$$P_{\mathbb{Z}_2}^1(S^1) = S^1 \times \mathbb{Z}_2, \quad P_{\mathbb{Z}_2}^2(S^1) = (S^1 \times \mathbb{Z}_2)/\mathbb{Z}_2, \quad (2.5)$$

where \mathbb{Z}_2 acts on $S^1 \times \mathbb{Z}_2$ by the map $(w, \pm 1) \mapsto (-w, \mp 1)$. The bundle projection maps onto S^1 are given by $P_{\mathbb{Z}_2}^1(S^1) \rightarrow S^1 \times \mathbb{Z}_2$, $(z, \pm) \mapsto \pm z$, and $P_{\mathbb{Z}_2}^2(S^1) \rightarrow (S^1 \times \mathbb{Z}_2)/\mathbb{Z}_2$, $(w, \pm) \mapsto w^2$. The first total space of the bundle is not connected. The second total space is connected, since $(S^1 \times \mathbb{Z}_2)/\mathbb{Z}_2 = \{[w, \varepsilon]\}$, with $[w, \varepsilon] = \{(w, \varepsilon), (-w, -\varepsilon)\}$. The properties of (and terminology for) the two Spin structures on the circle are summarized in the following table.

Trivial	bounding	non-supersymmetric	Neveu-Schwarz	even	periodic
Nontrivial	nonbounding	supersymmetric	Ramond	odd	antiperiodic

Spin structures on Riemann surfaces. Let Σ_g be a closed connected and orientable Riemann surface of genus g . Let $S^1 \xrightarrow{i} U(\Sigma_g) \xrightarrow{\pi} \Sigma_g$ be the unit tangent bundle (i.e. the sphere bundle of the tangent bundle). The cohomological definition of a Spin structure says that there is a cohomology class $\xi \in H^1(\Sigma_g; \mathbb{Z}_2)$ that restricts to a generator of $H^1(S^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$ on each fiber S^1 . The short exact sequence

$$0 \longrightarrow H^1(\Sigma_g; \mathbb{Z}_2) \xrightarrow{\pi^*} H^1(U(\Sigma_g); \mathbb{Z}_2) \xrightarrow{i^*} H^1(S^1; \mathbb{Z}_2) \longrightarrow 0 \quad (2.6)$$

establishes that the set $\text{Spin}(\Sigma_g)$ of Spin structures on Σ_g is in one-to-one correspondence with $H^1(\Sigma_g; \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$ [8]. The correspondence is not a group isomorphism and $\text{Spin}(\Sigma_g)$ is a nontrivial coset of $H^1(\Sigma_g; \mathbb{Z}_2)$ in $H^1(U(\Sigma_g); \mathbb{Z}_2)$. Let ξ_1, \dots, ξ_{2g} be a basis for $H^1(\Sigma_g; \mathbb{Z})$ and let z denote the extra element in $H^1(U(\Sigma_g); \mathbb{Z})$. A basis for $H^1(U(\Sigma_g); \mathbb{Z})$ is given by $\xi_1, \dots, \xi_{2g}, z$. Then the mod 2 reduction $r_2(z) \in H^1(U(\Sigma_g); \mathbb{Z}_2)$ is a particular Spin structure and the set of Spin structures is given by

$$\text{Spin}(\Sigma_g) = \left\{ \sum_{i=1}^{2g} x_i r_2(\xi_i) + r_2(z) \mid \text{all } x_i \in \mathbb{Z}_2 \right\}. \quad (2.7)$$

Spin structures on spheres. There is a unique Spin structure on the n -sphere S^n , $n > 1$, given by [45]

$$\text{Spin}(n+1) \rightarrow SO(n+1) \rightarrow S^n. \quad (2.8)$$

Spin structures on projective spaces. Consider the projective space $\mathbb{K}P^n$ over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or \mathbb{H} . The total Stiefel-Whitney class is

$$w(\mathbb{K}P^n) = 1 + w_1 + w_2 + \dots = (1 + x)^{n+1}, \quad (2.9)$$

where x is the generator of $H^*(\mathbb{K}P^n; \mathbb{Z}_2)$ with $\dim_{\mathbb{K}} x = 1$.

$\circ \mathbb{K} = \mathbb{R}$: Requiring $w_1 = 0 = w_2$ is equivalent to $n + 1 \equiv \frac{1}{2}n(n + 1) \equiv 0 \pmod{2}$. This means that $\mathbb{R}P^n$ is Spin iff $n \equiv 3 \pmod{4}$. Consider the latter case, $\mathbb{R}P^{2k+1}$: These real projective spaces are orientable with fundamental group (see section 2.2)

$$\pi_1(\mathbb{R}P^{2k+1}) = \begin{cases} \mathbb{Z}_2, & \text{for } k \text{ odd,} \\ \mathbb{Z}_2 \times \mathbb{Z}_2, & \text{for } k \text{ even,} \end{cases} \quad (2.10)$$

so that

1. $\mathbb{R}P^{4l+1}$ has $w_1 = 0$ and $w_2 \neq 0$ for $l = 1, 2, 3, \dots$.
2. $\mathbb{R}P^{4l-1}$ has $w_1 = 0$ and $w_2 = 0$ and $\pi_1 = \mathbb{Z}_2$, so has two inequivalent Spin structures

$$\pi^\pm : \text{Spin}(4l)/\mathbb{Z}_2^\pm \rightarrow \text{SO}(4l)/\mathbb{Z}_2, \quad (2.11)$$

where π^\pm are the obvious projections and the action of $\text{Spin}(4l - 1)$ in $\text{Spin}(4l)/\mathbb{Z}_2^\pm$ is obtained from the natural action of $\text{Spin}(4l - 1)$ in $\text{Spin}(4l)$ by passing to the quotient. The two inequivalent Spin structures are related by an orientation-reversing isometry.

$\circ \mathbb{K} = \mathbb{C}$: Requiring $w_1 = 0 = w_2$ is equivalent to $n + 1 \equiv 0 \pmod{2}$. This means that $\mathbb{C}P^n$ is Spin iff n is odd.

$\circ \mathbb{K} = \mathbb{H}$: The condition is vacuous for dimension reasons, so that the quaternionic projective space $\mathbb{H}P^n$ is Spin for all n .

Spin structures on Lie groups. Let G be a simply connected Lie group with a left invariant metric. The elements of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ can be regarded as left invariant vector fields. The choice of an oriented orthonormal basis of \mathfrak{g} gives a trivialization of the frame bundle $P_{\text{SO}}(G) = G \times \text{SO}(n)$. The unique Spin structure can be written as $P_{\text{Spin}}(G) = G \times \text{Spin}(n)$, with $\varphi = \text{id} \times \rho$ in (1.2). Spinor fields on the group G are then just maps $G \rightarrow \Delta_n$, corresponding to Spin representations Δ_n of the group G .

If G is not simply connected then the situation is different. The group manifold $\text{SO}(n)$ is not simply connected and so has two distinct Spin structures. Corresponding to the (trivial) principal frame bundle $P(\text{SO}(n)) = \text{SO}(n) \times \text{SO}(N)$, $N = \frac{1}{2}n(n - 1)$, are two coverings

$$\begin{aligned} Q_1(\text{SO}(n)) &= \text{SO}(n) \times \text{Spin}(N) \\ Q_2(\text{SO}(n)) &= (\text{Spin}(n) \times \text{Spin}(N)) / \mathbb{Z}_2, \end{aligned} \quad (2.12)$$

where \mathbb{Z}_2 acts on $\text{Spin}(n) \times \text{Spin}(N)$ by the map $(g, h) \mapsto (-g, -h)$.

Spin structures on Ricci-flat n -dimensional Spin manifolds. We review the description in [118] (and in more precise form in [128]). Consider a Ricci-flat n -manifold M . By the Cheeger-Gromoll theorem [41], M can be expressed as $M = \widetilde{M}/\Gamma$, where \widetilde{M} is the universal Riemannian cover of M and Γ is a finite group of isometries acting freely on \widetilde{M} . Let $f \in \Gamma$ and \tilde{f} be its lift to a bundle automorphism $\tilde{f} : P_{\text{SO}}(\widetilde{M}) \rightarrow P_{\text{SO}}(\widetilde{M})$ of $P_{\text{SO}}(\widetilde{M})$, and let \hat{f} be the Spin lift

$$\hat{f} : P_{\text{Spin}}(\widetilde{M}) \rightarrow P_{\text{Spin}}(\widetilde{M}) \quad (2.13)$$

covering \tilde{f} . Denote by $-\hat{f}$ the composite of \hat{f} with -1 , where -1 is the action by $-1 \in P_{\text{Spin}}(M)$ on $P_{\text{Spin}}(\widetilde{M})$. Then $P_{\text{Spin}}(\widetilde{M})/\Gamma$ is a principal $\text{Spin}(n)$ bundle over $\widetilde{M}/\Gamma = M$, and so is a Spin structure over M . There is an ambiguity in the way Γ acts on $\text{Spin}(\widetilde{M})$: Each f of even order can be represented by either \hat{f} or by $-\hat{f}$ (similarly for order p : $\hat{f}^p = 1$). These various actions by Γ produce different Spin structures when the quotient $\text{Spin}(\widetilde{M})/\Gamma$ is taken. The resulting Spin structures are in one-to-one correspondence with $\text{Hom}(\Gamma, \mathbb{Z}_2) = H^1(M; \mathbb{Z}_2)$. Consider the following examples [118].

Example 1: G_2 -manifolds. Let M be a Joyce manifold, i.e. a compact 7-manifold with holonomy group G_2 and fundamental group isomorphic to \mathbb{Z}_2 . So $M = \widetilde{M}/\mathbb{Z}_2$, with \mathbb{Z}_2 generated by an involution f . Then

\widetilde{M} has a unique Spin structure $P_{\text{Spin}}(\widetilde{M})$ but M has two distinct Spin structures $P_{\text{Spin}}(\widetilde{M})/\{1, \hat{f}\}$ and $P_{\text{Spin}}(\widetilde{M})/\{1, -\hat{f}\}$.

Example 2: Calabi-Yau threefolds. Let M be a Calabi-Yau 3-fold of the form $\widetilde{M}/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, with $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by a pair of commuting involutions (a, b) . Then M has *four* distinct Spin structures, given by $P_{\text{Spin}}(\widetilde{M})/\Gamma'$, with Γ' one of the finite groups $\{1, \hat{a}, \hat{b}, \hat{a}\hat{b}\}$, $\{1, -\hat{a}, \hat{b}, -\hat{a}\hat{b}\}$, $\{1, \hat{a}, -\hat{b}, \hat{a}\hat{b}\}$, $\{1, -\hat{a}, -\hat{b}, \hat{a}\hat{b}\}$.

2.2 Spin structures and the fundamental group

We have seen that the main source of multiple Spin structures is spaces with nontrivial fundamental group. We study this in more detail here. We start with (classes of) examples and then consider the general case.

Flat manifolds. Flat manifolds are of course solutions to both M-theory in eleven dimensions and to type IIA string theory in ten dimensions.

1. *The n -torus.* The n -torus T^n admits 2^n Spin structures [72]. Consider $T^n = \mathbb{R}^n/\Gamma$ with Γ a lattice in \mathbb{R}^n . Let b_1, \dots, b_n be a \mathbb{Z} -basis for Γ and b_1^*, \dots, b_n^* be a dual basis for the dual lattice Γ^* . Spin structures on T^n can be classified by n -tuples $(\delta_1, \dots, \delta_n)$ where each $\delta_j \in \{0, 1\}$ indicates the twist of the Spin structure in the direction b_j .

2. *Torus bundles over flat manifolds.* Such manifolds are classical solutions to supergravity. There are many examples of these toral extensions which are Spin. However, this is not automatic and there also exist examples which are not Spin [17].

3. *Flat manifolds with cyclic holonomy.* Let M^n be an oriented n -dimensional flat Riemannian manifold. The fundamental group $\Gamma = \pi_1(M^n)$ determines a short exact sequence $0 \rightarrow \mathbb{Z}^n \rightarrow \Gamma \xrightarrow{p} F \rightarrow 0$, where \mathbb{Z}^n is a torsion-free abelian group of rank n and F is a finite group which is isomorphic to the holonomy group of M^n . The universal cover of M^n is \mathbb{R}^n , so that Γ is isomorphic to a discrete cocompact subgroup of the isometry group $\text{Isom}(\mathbb{R}^n) = \mathbb{R}^n \ltimes SO(n)$. The existence of a Spin structure on M^n is equivalent to the existence of a homomorphism $\epsilon : \Gamma \rightarrow \text{Spin}(n)$ such that $\rho\epsilon = p$, where $\rho : \text{Spin}(n) \rightarrow SO(n)$ is the covering map. Any flat manifold with holonomy of odd order admits a Spin structure [161]. In fact, every finite group is the holonomy group of a Spin flat manifold [48]. If the fundamental groups Γ_i , $i = 1, 2$ of two flat oriented n -manifolds M_i are isomorphic then M_1 has a Spin structure iff M_2 has a Spin structure [93]. Multiple Spin structures are possible here. For example, if Γ is of the so-called diagonal type and M^n has holonomy \mathbb{Z}_2 then there are 2^n Spin structures (as in the case of the n -torus) [119].

2.2.1 Spherical space forms

A classical result of Hopf characterizes a spherical space form as follows: A Riemannian manifold M of dimension $n \geq 2$ is a connected complete manifold of positive constant curvature iff M is isometric to S^n/Γ , where Γ is a finite subgroup of $O(n+1)$ which acts freely on the sphere S^n . Any fixed-point-free map $g : S^n \rightarrow S^n$ is homotopic to the antipodal map. Hence $\deg(g) = (-1)^{n+1}$. Thus if n is even, the composite of two fixed-point free maps has a fixed point. This means that the only group which acts freely on an even-dimensional sphere is \mathbb{Z}_2 , so the only spherical space forms in even dimensions are the sphere itself S^{2m} , corresponding to the trivial element in \mathbb{Z}_2 , and the real projective space $\mathbb{R}P^{2m}$, corresponding to the nontrivial element in \mathbb{Z}_2 . However, the case n odd is much more interesting.

The main examples of spherical space forms are Clifford-Klein manifolds. These are complete Riemannian manifolds with constant sectional curvature equal to $+1$, and are of the form S^n/Γ , where Γ is a finite group acting freely and orthogonally on S^n , as described above. Equivalently, they are given by an orthogonal representation $\varrho : \Gamma \rightarrow O(n+1)$ with $\varrho(g)$ having no $+1$ eigenvalue for all nontrivial elements $g \in \Gamma$. The representation ϱ is conjugate to a unitary fixed-point-free representation $\varrho : \Gamma \rightarrow U(k) \subseteq SO(2k) \subset O(2k)$. The classification of Clifford-Klein manifolds $M(\Gamma, \varrho)$ is thus a completely algebraic question in group

representation theory, whose solution is given completely by Wolf [167]. More precisely, the classification of spherical space forms can be reduced to the determination of all finite groups having fixed-point-free real orthogonal representations and hence free unitary representations.

Since S^n is a universal Γ -covering space of M^n , and since $\pi_1(M^n) = \Gamma$ and $\pi_i(M^n) \cong \pi_i(S^n)$ for $i \neq n$, the integral cohomology groups of M^n are

$$H^i(M^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0 \text{ or } i = n, \\ H^i(\Gamma; \mathbb{Z}), & \text{if } 0 < i < n, \\ 0, & \text{otherwise.} \end{cases} \quad (2.14)$$

If Γ acts freely on S^{2n-1} then Γ has a free resolution of length (or period) $2n$ [47]. A free resolution of period n of Γ is an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\mu} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \quad (2.15)$$

of $\mathbb{Z}\Gamma$ -modules with the F_i finitely generated and free, with Γ acting trivially on the two \mathbb{Z} terms. The fact that Γ has a free resolution of period $2n$ implies that $H^{2n+1}(\Gamma; \mathbb{Z}) = \mathbb{Z}/|\Gamma|$, which gives that the abelian subgroups of Γ are cyclic [47]. For example:

1. The cyclic groups \mathbb{Z}_n act freely and orthogonally on S^1 .
2. The binary dihedral groups $Q(4k) = \{x, y \mid x^{2k} = 1, x^k = y^2, yxy^{-1} = x^{-1}\}$ are subgroups of the unit quaternions S^3 and so act freely and orthogonally on S^3 via quaternionic multiplication. When k is a power of 2, $Q(4k)$ is a generalized quaternion group.

A finite group Γ is called a *spherical space form group* if Γ satisfies either of the two equivalent conditions [47]:

- (i). The cohomology of Γ is periodic.
- (ii). The group Γ acts without fixed points on some sphere.

Let Γ be a nontrivial finite group acting linearly and freely on S^n . Then

1. If $n = 2m$, Γ is isomorphic to \mathbb{Z}_2 and the only nontrivial even-dimensional spherical forms are the real projective spaces $\mathbb{R}P^{2m}$. Let $M = \mathbb{R}P^m$, so that $\pi_1(M) = \mathbb{Z}_2 = \{\pm 1\}$. Then the covering space is $\widetilde{M} = S^m$ and the deck group action is the usual action of $\{\pm 1\}$ on S^m given by multiplication. The nontrivial element -1 acts as the antipodal map and the trivial element $+1$ acts as the identity. The orthogonal representation of the fundamental group

$$\varrho : \pi_1(\mathbb{R}P^m) = \mathbb{Z}_2 \rightarrow O(1) = \mathbb{Z}_2 \quad (2.16)$$

is given by $\varrho(\pm 1) = \pm 1$.

2. If $n = 2m + 1$, then

- if Γ is *abelian*, then Γ is cyclic; these correspond to lens spaces. The cyclic group \mathbb{Z}_n , the group of n th roots of unity, acts on the unit sphere S^{2k-1} in \mathbb{C}^k by complex multiplication. The lens space is the quotient manifold

$$L^{2k-1} := S^{2k-1} / \mathbb{Z}_n. \quad (2.17)$$

The irreducible representations of \mathbb{Z}_n are parametrized by $\{\varrho_s\}$, $0 \leq s < n$, where $\varrho_s(\lambda) = \lambda^s$.

- if Γ is *nonabelian*, then it has the following equivalent properties:
 1. all the abelian subgroups of Γ are cyclic;
 2. the p -Sylow subgroups Γ_p of Γ are of two types: either all of them are cyclic, or they are cyclic for $p \neq 2$ and generalized quaternion groups for $p = 2$. The latter corresponds to the quaternionic spherical space form S^{2m+1}/Q_r . The canonical inclusion $i_p : \Gamma_p \rightarrow \Gamma$ induces a ring homomorphism $i_p^* : RO(\Gamma) \rightarrow RO(\Gamma_p)$, so that Γ_p acts freely on S^n and the resulting quotient $M_p^n = S^n / \Gamma_p$ is the spherical p -form associated to M^n , and $i_p : M_p^n \rightarrow M^n$ is a covering fibration.

There is a difference between abelian and nonabelian fundamental groups.

Nonabelian fundamental group. A manifold can have a *nonabelian* fundamental group yet have only one Spin structure. For example [118]: Γ is the nonabelian group of order 12 generated by a, b, c satisfying

$$a^2 = b^2 = c^3 = 1, \quad ab = ba, \quad cac^{-1} = b, \quad cbc^{-1} = ab. \quad (2.18)$$

The relation $\hat{c} \hat{a} \hat{c}^{-1} = \hat{b}$ shows that \hat{a} can be replaced by $-\hat{a}$ only if \hat{b} is replaced by $-\hat{b}$, but this is inconsistent with $\hat{c} \hat{b} \hat{c}^{-1} = \hat{a}$. Therefore, $M = \widetilde{M}/\Gamma$ has only *one* Spin structure in this case.

Group representation ring of \mathbb{Z}_n . The integers modulo n , $\mathbb{Z}_n := \{\lambda \in \mathbb{C} \mid \lambda^n = 1\}$, is the group of n th roots of unity. As above, let $\varrho_s(\lambda) := \lambda^s$. The $\{\varrho_s\}$ for $s \in \mathbb{Z}_n$ parametrize the irreducible unitary representations of \mathbb{Z}_n , so the group representation ring $RU(\mathbb{Z}_n) = \oplus_{s \in \mathbb{Z}_n} \varrho_s \cdot \mathbb{Z}$ is the free abelian group generated by these elements. The ring structure is given by $\varrho_s \cdot \varrho_t = \varrho_{s+t}$. Since $\varrho_1^n = 1$, then

$$RU(\mathbb{Z}_n) = \mathbb{Z}[\varrho_1]/(\varrho_1^n = 1). \quad (2.19)$$

Generalized lens spaces. The cyclic group \mathbb{Z}_n is a spherical space form group. Let $\vec{a} = (a_1, \dots, a_k)$ be a collection of integers and let $\tau' = \oplus_{\nu} \varrho_{a_{\nu}}$. Then τ' is fixed-point-free iff all the a_{ν} are coprime to n . The generalized lens spaces

$$L(n; \vec{a}) = M(\mathbb{Z}_n, \tau') = S^{2k-1}/\varrho(\mathbb{Z}_n) \quad (2.20)$$

are the spherical space forms with cyclic fundamental groups. If $k = 1$ or $k = 3$ then TM is trivial so that M is always Spin. For $k \geq 5$, [78] ϱ can be lifted to $\text{Spin}(2k)$, so that $M(\Gamma, \varrho)$ is Spin, if $|\Gamma|$ is odd, or if $|\Gamma|$ is even and k is even. Note that ϱ automatically lifts to $\text{Spin}(2k)$ if $|\Gamma|$ is even and if k is odd.

Now consider Spin structures on the space bounding a lens space. Such an example is important for the M-theory partition function. In general, if Z^{12} is a Riemannian Spin manifold, then a Spin structure on Z^{12} induces a Spin structure on $Y^{11} = \partial Z^{12}$. The frame bundle $P_{SO}Y^{11}$ can be considered as a subbundle of the frame bundle $P_{SO}Z^{12}$ restricted to the boundary by adding the unit normal vector to the set of orthonormal frames of the former. The inverse image of $P_{SO}Y^{11}$ under the covering map $P \rightarrow P_{SO}Z^{12}$ defines a Spin structure on Y^{11} . Any Spin structure on $Z^{12}(p, \vec{a})$ is obtained by twisting the canonical Spin^c structure σ_{can}^c (see section 3.1) by the square root $K^{1/2}$ of the canonical line bundle K of $Z^{12}(p, \vec{a})$. The canonical line bundle K corresponds to the representation (cf. [75])

$$a_1 + a_2 + \dots + a_n \in \mathbb{Z}_p \cong R(\mathbb{Z}_p, U(1)). \quad (2.21)$$

The square root $K^{1/2}$ corresponds to an element $c \in \mathbb{Z}_p \cong R(\mathbb{Z}_p, U(1))$ satisfying $2c \equiv a_1 + a_2 + \dots + a_n \pmod{p}$. Therefore,

- $Z^{12}(p, \vec{a})$ is Spin iff p is odd, or p and the sum $a_1 + \dots + a_n$ are both even.
- If p is odd then the Spin structure $\sigma = c$ is unique.
- If p and $a_1 + \dots + a_n$ are both even then there are two Spin structures $\sigma_1 = c$ and $\sigma_2 = c + p/2$.

Let $\vec{a} \in \mathbb{Z}^n$ be an n -tuple of integers coprime to an integer p . Then the following properties hold [75]:

- (1) The lens space $L(p; \vec{a})$ admits a Spin structure iff p is odd, or p is even and $\sum_{k=1}^n a_k$ is even.
- (2) In case the conditions in (1) hold for $(p; \vec{a}) \in \mathbb{Z}^{n+1}$ there is a one-to-one correspondence between the set of all Spin structures on $L(p; \vec{a})$ and the set of all elements $c \in \mathbb{Z}_p$ satisfying

$$2c = a_1 + \dots + a_n \in \mathbb{Z}_p. \quad (2.22)$$

Killing spinors in non-simply connected case. Supersymmetry involves (generalized) Killing or parallel spinors. Let (M, g) be a complete n -dimensional Riemannian Spin manifold, and let SM be the Spin bundle of M and ψ a smooth section of SM . Then ψ is a Killing spinor if

$$\nabla_X \psi = \alpha X \cdot \psi, \quad \forall X \in \Gamma(TM), \quad (2.23)$$

where ∇ is the Levi-Civita connection of g and $\mathbf{X} \cdot \psi$ denotes the Clifford multiplication of \mathbf{X} and ψ . We say that ψ is *imaginary* when the Killing constant $\alpha \in \text{Im}(\mathbb{C}^*)$, ψ is *parallel* if $\alpha = 0$ and ψ is real if $\alpha \in \text{Re}(\mathbb{C}^*)$. Now let (M^n, g) be a complete Riemannian Spin manifold with nontrivial fundamental group, $\pi_1(M) \neq 0$, and a nontrivial Killing spinor with $\alpha > 0$ or $\alpha < 0$. If $n = 2m + 1$, $m \geq 2$ even, then there are two possibilities [162]:

- (1) M is a Spin spherical space form.
- (2) M is Sasakian-Einstein with $\text{Hol}(\overline{g}) = SU(m + 1)$.

The above spaces are important in classical considerations in M-theory (and we will consider the partition function later in this paper). Superconformal Chern-Simons theories in three dimensions are obtained by taking a factor in the eleven-dimensional manifold to be an eight-dimensional transverse cone with base the quotient S^7/Γ , with $\Gamma < SO(8)$ a finite group acting freely on S^7 such that the quotient is Spin and the space of Killing spinors has dimension N . The classification for $N > 3$ is obtained in [49]. Let S^{2m-1}/Γ be a spherical space form with Spin structure given by $\epsilon : \Gamma \rightarrow \text{Spin}(2m)$. Then the dimension of the space of Killing spinors with Killing constant $\alpha = \pm \frac{1}{2}$ is given by [18] $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi^\pm(\epsilon(\gamma))$, where χ^\pm are the characters of the two irreducible half-spin representations of $\text{Spin}(2m)$.

Homology and higher connectedness. If π_1 is the first nonvanishing homotopy group of a manifold M , then by the Hurewicz theorem the first homology group is isomorphic to the abelianization of π_1 . Beyond that, n -connected manifolds, with $n \geq 2$, carry unique Spin structures. Examples include: homotopy spheres (e.g. squashed spheres), Stiefel manifolds, and simply-connected Lie groups. The case $n = 2$ will be considered in section 6.3 in connection to Spin^c structures. The case $n = 3$ is considered in [150] in connection to String structures.

2.2.2 Holonomy and flat vector bundles

Let \widetilde{M} be the universal cover of M . The fundamental group $\pi_1(M)$ of M acts on \widetilde{M} by deck transformations $\tilde{x} \rightarrow g \cdot \tilde{x}$, for $\tilde{x} \in \widetilde{M}$ and $g \in \pi_1(M)$. There is a natural identification between representations ϱ of the fundamental group and flat vector bundles. Start with a vector bundle V and a Riemannian connection on V which has zero curvature. Let \mathbb{F}_0^k be the fiber of V over the basepoint x_0 , and where \mathbb{F} is \mathbb{R} or \mathbb{C} . Parallel transport around a closed path γ defines the holonomy representation of $\pi_1(M)$. The bundle V is naturally isomorphic to the associated flat vector bundle V_ϱ , given by

$$V_\varrho := \left(\widetilde{M} \times_{\varrho} \mathbb{F}^k \right) / \sim, \quad (2.24)$$

where $(\tilde{x}, v) \sim (g \cdot \tilde{x}, \varrho(g)v)$ for all $g \in \pi_1(M)$. The associated connection agrees with the original connection. Denote by $\mathcal{O}_{\mathbb{F}}^k$ the trivial real or complex vector bundle with fiber \mathbb{F}^k . Define a flat connection on this bundle by $\widetilde{\nabla} \tilde{f} = d\tilde{f}$, where \tilde{f} is a smooth section of $\mathcal{O}_{\mathbb{F}}^k$. The section descends to a section on V_ϱ and also $\widetilde{\nabla}$ descends to a connection on V_ϱ with zero curvature. For example, consider the case around equation (2.16). The associated vector bundle is given by $L := S^m \times \mathbb{R} / \sim$, where the equivalence is given by $(\xi, x) \sim (-\xi, -x)$. The bundle L is a real line bundle which is isomorphic to the classifying line bundle over $\mathbb{R}P^m$. Next, considering the lens space (2.17), let L_s be the complex line bundle defined by the representation ϱ_s . Then $L_s = S^{2k-1} \times \mathbb{C} / \sim$, where the equivalence is given by $(\xi, z) \sim (\lambda \xi, \lambda^s z)$ for $\lambda \in \mathbb{Z}_n$, $\xi \in S^{2k-1}$, and $z \in \mathbb{C}$.

Principal bundles with finite structure group. Let Γ be a finite group. A Γ -structure on a connected manifold X is a principal Γ bundle P over X . The holonomy of the bundle P defines a group homomorphism from the fundamental group $\pi_1(X)$ of X to Γ . On the other hand, let Γ act without fixed points on a compact manifold P . Then $\Gamma \curvearrowright P \rightarrow P/\Gamma$ is a principal Γ bundle over $X = P/\Gamma$. The study of principal Γ bundles is equivalent to the study of fixed-point-free Γ actions (see [79]).

Lattices of Lie groups. This is yet another rich source of Spin structures. Let $\Gamma \subset G$ be a lattice in a Lie group G . Spin structures on $M = G/\Gamma$ correspond to homomorphisms $\epsilon : \Gamma \rightarrow \mathbb{Z}_2 = \{-1, 1\}$. The corresponding Spin structure is given by

$$P_{\text{Spin}, \epsilon}(G/\Gamma) = G \times_{\Gamma} \text{Spin}(n), \quad (2.25)$$

where $g_0 \in \Gamma$ acts on G by left multiplication and on $\text{Spin}(n)$ by multiplication with the central element $\epsilon(g_0)$. Spinor fields on M can be identified with ϵ -equivariant maps to the Spinor module $\psi : G \rightarrow \Delta_n$, i.e. $\psi(g_0 g) = \epsilon(g_0) \psi(g)$ for all $g \in G$, $g_0 \in \Gamma$.

2.3 Effect of Spin structures on geometric and analytical entities

The spectrum. Consider the example of the circle S^1 from section 2.1. The Dirac operator is simply $D = i \frac{d}{dt}$, with parameter t . Spinors on the circle corresponding to the trivial Spin structure are complex valued functions. The spectrum corresponding to the trivial Spin structure consists of the eigenvalues $\lambda_k = k$ with eigenfunctions $t \mapsto e^{-ikt}$, $t \in \mathbb{Z}$. On the other hand, spinors corresponding to the nontrivial Spin structure are 2π -anti-periodic complex-valued functions on \mathbb{R} , with eigenvalues $\lambda_k = k + \frac{1}{2}$, $k \in \mathbb{Z}$, with eigenfunction $t \mapsto e^{-i(k+\frac{1}{2})t}$. This example illustrates dependence of the spectrum of the Dirac operator on the choice of Spin structure. This phenomenon persists in higher dimensions as well. Many examples, such as spherical space forms, can be found in [19].

The noncommutative case. In the presence of a Neveu-Schwarz B -field, spacetime can become noncommutative (see [152]). Spin structures on noncommutative spaces are delicate. They are closely related to the reality structure J appearing in spectral triples, but further require invoking spectral properties of the Dirac operator. The noncommutative two-sphere, like the classical one, has one admissible (real) Spin structure. There are a priori others, but these are ruled out by spectral properties of the Dirac operator [133]. The noncommutative two-torus has, similarly to the classical case, four inequivalent Spin structures. The spectra of the corresponding Dirac operators depend on the Spin structure [134]. However, interestingly, there are examples of spherical space forms where different Spin structures give rise to the *same* spectral action [114].

Eta invariants. As a result of asymmetry in the spectrum, the values of the η -invariants depend on the choice of Spin structures. For example, for the real projective space $\mathbb{R}P^n$, $n \equiv 3 \pmod{4}$, admitting two Spin structures, the η -invariant for the Dirac operator is given by $\eta = \pm 2^{-m}$, $n = 2m - 1$, where the sign depends on the Spin structure chosen [19]. The differences are more drastic in the case of Bieberbach manifolds $B^n = T^n/\Gamma$; for example, for $\Gamma = \mathbb{Z}_4$, B^n has 4 Spin structures and the values of the eta invariant are: $\eta = 0$ for two of them, $\eta = \frac{3}{2}$ for one, and $\eta = -\frac{1}{2}$ for the fourth Spin structure [137].

Harmonic spinors. The space of solutions of a spinor field equation usually depends on the Spin structure. For example, the dimension of the kernel of the Dirac operator corresponding to the two Spin structures on the circle are: one for the trivial Spin structure and zero for the nontrivial Spin structure on S^1 . The case of harmonic spinors – i.e. the ‘bare’ Killing spinors – is explained extensively in [94] [20].

Killing and parallel spinors. In the case of multiple Spin structures, extending a local parallel spinor depends on the choice of Spin structure [118]. There are compact irreducible Ricci-flat manifolds with nongeneric holonomy in dimension 4 and 8 which admit two Spin structures, each of which carries a parallel spinor [128]; an example is a Calabi-Yau fourfold with holonomy $SU(4) \rtimes \mathbb{Z}_2$ obtained via an involution as a complete intersection in $\mathbb{C}P^9$. Specifying a Spin structure as part of the data defining a supergravity background was highlighted in [67]. For example, it was shown that the same geometry of four-dimensional anti-de Sitter space times a lens space S^7/\mathbb{Z}_4 preserves a different amount of supersymmetry depending on the choice of Spin structure. Such a phenomenon is expected to persist for more general spherical space forms.

Spin holonomy and preferred Spin structure(s). Consider an $SO(n)$ bundle P over an oriented Spin manifold (M, g) with Levi-Civita connection ∇ . Choose a Spin structure (P_{Spin}, ρ) on M , so that ∇ is lifted to a Spin connection ∇^s on the Spin bundle $S = P_{\text{Spin}} \times_{\text{Spin}(n)} \Delta_n$ corresponding to P_{Spin} . Here ρ is the projection $\rho : \text{Spin}(n) \rightarrow SO(n)$. The holonomy group $\text{Hol}(\nabla^s)$ is a subgroup of $\text{Spin}(n)$, and coincides with $\text{Hol}(g)$ under the projection map ρ . The projection $\rho : \text{Hol}(\nabla^s) \rightarrow \text{Hol}(g)$ is either an isomorphism or a double cover. This depends on the choice of Spin structure [98]. However, if M is simply connected then both $\text{Hol}(g)$ and $\text{Hol}(\nabla^s)$ are connected, so that the latter is the identity component of $\rho^{-1}(\text{Hol}(g))$ in $\text{Spin}(n)$. Therefore, for simply connected Spin manifolds, the Riemannian holonomy groups and the Spin holonomy groups coincide.

Let M be an n -dimensional manifold, $n \geq 3$, that admits a G -structure P_G with G a connected, simply connected subgroup of $SO(n)$. Then M is Spin and has a natural Spin structure P_{Spin} induced by P_G . Since all of the Ricci-flat holonomy groups, $SU(n)$, $Sp(n)$, G_2 , and $\text{Spin}(7)$ are connected and simply connected, this gives the following result [98]: Let (M, g) be a Riemannian manifold and suppose that $\text{Hol}(g)$ is one of the Ricci-flat holonomy groups above. Then M is Spin, with a *preferred Spin structure*. With this Spin structure, the spaces of parallel spinors on M are nonzero. Thus, an irreducible metric has one of the Ricci-flat holonomy groups iff it admits a nonzero constant spinor.

Effect of orientation-reversal on spinors on Y^{11} . Let Y^{11} be a Riemannian manifold and let $P_{SO(11)}$ be the $SO(11)$ -principal bundle of oriented orthonormal frames. Let $P_{\text{Spin}(11)} \rightarrow Y^{11}$ and $\sigma : P_{\text{Spin}(11)} \times_{\text{Spin}(11)} SO(11) \cong P_{SO(11)}$ represent the Spin structure. Now let $P_{SO(11)}^{\text{op}} \rightarrow Y^{11}$ be the $SO(11)$ -principal bundle of oriented orthonormal frames of Y_{op}^{11} , where the latter refers to the manifold Y^{11} with the reversed orientation. Define an isomorphism of $SO(11)$ -principal bundles $\theta : P_{\text{Spin}(11)} \rightarrow P_{\text{Spin}(11)}^{\text{op}}$ which maps the frame (Y_1, \dots, Y_{11}) to $(-Y_1, \dots, -Y_{11})$. Then the opposite Spin structure is given by $P_{\text{Spin}(11)} \rightarrow Y^{11}$ and $P_{\text{Spin}(11)} \times_{\text{Spin}(11)} SO(11) \xrightarrow{\sigma} P_{SO(11)} \xrightarrow{\theta} P_{SO(11)}^{\text{op}}$.

Effect of orientation-reversal on spinors on X^{10} . Let X^{10} be a Riemannian manifold and let $P_{SO(10)}$ be the $SO(10)$ -principal bundle of oriented orthonormal frames. Let $P_{\text{Spin}(10)} \rightarrow X^{10}$ and $\sigma : P_{\text{Spin}(10)} \rightarrow X^{10} \times_{\text{Spin}(10)} SO(10) \cong P_{SO(10)}$ represent the Spin structure. Let $P_{SO(10)}^{\text{op}} \rightarrow X^{10}$ be the $SO(10)$ -principal bundle of oriented orthonormal frames of X_{op}^{10} , where the latter refers to the manifold X^{10} with the reversed orientation. The image $E_1 \subset O(10)$ of an element $e_1 \in \mathbb{R}^{10} \subset \text{Pin}(10) \subset \mathcal{Cl}(\mathbb{R}^{10})$ acts as $(x_1, x_2, \dots, x_{10}) \mapsto (x_1, -x_2, \dots, -x_{10})$. Define the map $\theta : P_{SO(10)} \rightarrow P_{SO(10)}^{\text{op}}$ by $(X_1, X_2, \dots, X_{10}) \mapsto (X_1, -X_2, \dots, -X_{10})$. This becomes an isomorphism of $SO(10)$ -principal bundles if the action of $SO(10)$ is twisted by E_1 , leading to $\tilde{P}_{SO(10)}$, the twisted bundle $P_{SO(10)}$. Let $P_{\text{Spin}(10)}^{\text{op}} \rightarrow Y^{11}$ be the $\text{Spin}(11)$ -principal bundle given by $P_{\text{Spin}(11)} \rightarrow Y^{11}$ with the $\text{Spin}(11)$ -action twisted by e_1 . As in [39], this gives a homomorphism of $SO(11)$ -principal bundles $P_{\text{Spin}(11)}^{\text{op}} \times_{\text{Spin}(11)} SO(11) \xrightarrow{\tilde{\sigma}} \tilde{P}_{SO(11)} \xrightarrow{\theta} P_{SO(11)}^{\text{op}}$, where $\tilde{\sigma}[s, v] := \sigma[sE_1, vE_1]$.

2.4 Circle actions and Spin structures

We now consider the situation in the setting of M-theory. We take Y^{11} to be a Riemannian eleven-dimensional manifold with metric g_Y , carrying a free isometric and geodesic action of the circle S^1 . Then the orbit space $X^{10} = Y^{11}/S^1$ is a manifold and there is a metric g_X on X^{10} such that the quotient map $\pi : Y^{11} \rightarrow X^{10}$ becomes a principal S^1 bundle and a Riemannian submersion.

The S^1 bundle has a unique connection 1-form $i\omega : TY^{11} \rightarrow i\mathbb{R}$ such that $\ker \omega|_y$ is orthogonal to the fibers with respect to g_Y for all $y \in Y^{11}$. The S^1 -action induces a Killing vector field v . The size of the fibers is measured by the dilaton field. There are two cases:

1. The dilaton ϕ is constant on X^{10} , i.e. the length $|v|$ is constant on Y^{11} , then the fibers of π are totally geodesic.

2. If the dilaton is not constant, then we have a nonconstant length function $e^\phi : X^{10} \rightarrow \mathbb{R}^+$. Rescaling the metric on Y^{11} is completely characterized by the connection 1-form $i\omega$, the fiber length $2\pi e^\phi$, and the metric g_X on X^{10} ; the Dirac operator can be expressed in terms of ω , e^ϕ , and g_X .

The S^1 -action on Y^{11} induces an S^1 -action on $P_{SO}(Y^{11})$. A Spin structure $\sigma_Y : P_{\text{Spin}}(Y^{11}) \rightarrow P_{SO}(Y^{11})$ is called (cf. [3])

1. *projectable* or *even* if this S^1 -action on $P_{SO}(Y^{11})$ lifts to $P_{\text{Spin}}(Y^{11})$.
2. *nonprojectable* or *odd* if the S^1 -action does not lift.

By a theorem of [89] all fibers of a Riemannian submersion with totally geodesic fibers are isometric, so there is a positive real number r such that all the fibers of π are isometric to $S_r^1 \hookrightarrow \mathbb{C}$, a circle of radius r in $\mathbb{C} \cong \mathbb{R}^2$ with its standard metric. There is the disk bundle $\pi_D : Z^{12} = Y^{11} \times_{S^1} \mathbb{D}^2 \rightarrow X^{10}$ of the associated complex line bundle $\pi_{\mathbb{C}} : \mathcal{L} = Y^{11} \times_{S^1} \mathbb{C} \rightarrow X^{10}$ to π . The disk $\mathbb{D}^2 \subset \mathbb{C}$ can be endowed with a metric such that $\partial\mathbb{D}^2$ is isometric to S_r^1 .

Let us consider some details. Denote by $\text{pr} : \text{Fr}(Y^{11}) \rightarrow Y^{11}$ the bundle of oriented frames, whose sections are the vierbeins, and by $p : P_{\text{Spin}}(Y) \rightarrow Y^{11}$ the principal Spin(11)-bundle over Y^{11} , with covering map $u : P_{\text{Spin}}(Y^{11}) \rightarrow \text{Fr}(Y^{11})$ corresponding to the double covering $\rho_{\text{Spin}} : \text{Spin}(11) \rightarrow SO(11)$. A given Spin structure is denoted $\sigma_Y = (P_{\text{Spin}}(Y^{11}), u)$. Let $\chi : S^1 \times Y^{11} \rightarrow Y^{11}$ be a smooth circle action. Then χ induces a circle action on $\text{Fr}(Y^{11})$ that can be viewed as a one-parameter family of maps $\chi_t : \text{Fr}(Y^{11}) \rightarrow \text{Fr}(Y^{11})$, $t \in \mathbb{R}$. The special values χ_0 and χ_1 coincide with the identity id_{Fr} on $\text{Fr}(Y^{11})$. Since $P_{\text{Spin}}(Y^{11})$ is the double cover of $\text{Fr}(Y^{11})$, χ_t lifts to $\tilde{\chi}_t : P_{\text{Spin}}(Y^{11}) \rightarrow P_{\text{Spin}}(Y^{11})$ with (cf. [10]):

1. *Even type*: $\tilde{\chi}_0 = \tilde{\chi}_1 = \text{id}_{P_{\text{Spin}}(Y^{11})}$, in which case $\tilde{\chi}_t$ induces a circle action on $P_{\text{Spin}}(Y^{11})$ which commutes with the right action of Spin(11) on $P_{\text{Spin}}(Y^{11})$ and is compatible with u . In this case χ is said to be of *even* type.
2. *Odd type*: $\tilde{\chi}_0 = \text{id}_{P_{\text{Spin}}(Y^{11})}$ and $\tilde{\chi}_1$ is multiplication by $-1 \in \mathbb{Z}_2 = \ker\{\text{Spin}(11) \rightarrow SO(11)\} \subset \text{Spin}(11)$, in which case $\tilde{\chi}_t$ induces an action of the connected double covering \hat{S}^1 of S^1 on $P_{\text{Spin}}(Y^{11})$ which, again, commutes with the right action of Spin(11) on $P_{\text{Spin}}(Y^{11})$ and is compatible with u . In this case χ is said to be of *odd* type.

Let $\pi_S : P_{\text{Spin}}(Y^{11}) \rightarrow P_{\text{Spin}}(Y^{11})/S^1$ and $\pi_{\text{Fr}} : \text{Fr}(Y^{11}) \rightarrow \text{Fr}(Y^{11})/S^1$ be the projections of the corresponding bundles to the quotient space. Note that there is an isomorphism $\pi_{\text{Fr}}^*(\text{Fr}(Y^{11})/S^1) \cong \text{Fr}(Y^{11})$.

The two situations for circle actions above are summarized in the following table

S^1 -action	Y^{11}	X^{10}
even	Spin	Spin
odd	Spin	Spin ^c

(2.26)

and will be treated separately and in detail in section 2.5 and section 2.6, respectively.

Cobordism of free circle actions. Let F_{11}^{Spin} denote the cobordism group of free circle actions on closed 11-dimensional Spin manifolds. This splits, according to whether the circle action is even or odd, as a direct sum

$$F_{11}^{\text{Spin}} = F_{11}^{\text{Spin, ev}} \oplus F_{11}^{\text{Spin, odd}}, \quad (2.27)$$

i.e. into bordism groups of free circle actions of even and odd type, respectively. The above discussion can be recast into cobordism language as follows (cf. [33]). For circle actions of even type we have the isomorphism $F_{11}^{\text{Spin, ev}} \cong \Omega_{10}^{\text{Spin}}(K(\mathbb{Z}, 2))$, mapping $[Y^{11}, \chi]$ to $[Y^{11}/S^1, f]$, where $f : Y^{11}/S^1 \rightarrow K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ classifies the complex line bundle \mathcal{L} associated with π . For circle actions of odd type we have the isomorphism $F_{11}^{\text{Spin, odd}} \cong \Omega_{10}^{\text{Spin}^c}$, mapping $[Y^{11}, \chi]$ to $[Y^{11}/S^1]$. See section 3.5 for details on Spin and Spin^c cobordism.

2.5 Projectable Spin structures

A projectable Spin structure $\sigma_Y : P_{\text{Spin}}(Y^{11}) \rightarrow P_{SO}(Y^{11})$ on Y^{11} induces a Spin structure on X^{10} as follows (see [3] [157] for the general formalism). Let the eleventh frame vector be $ve^{-\phi}$. The tangent bundle to the quotient which factorizes as $T(Y^{11})/S^1 = T(Y^{11}/S^1) \oplus \mathcal{O}$ is given as $T(Y^{11}/S^1) = P_{\text{Spin}}(Y^{11})/S^1 \times_{\text{Spin}(11)} \mathbb{R}^{11}$. The ten-dimensional frame bundle $P_{SO}(X^{10})$ can be identified with the quotient $P_{SO(10)}(Y^{11})/S^1$ and $\sigma_Y^{-1}(P_{SO(10)}(Y^{11})/S^1)$ is a Spin(10) bundle over X^{10} . Thus σ_Y induces the corresponding Spin structure on X^{10} .

The above Spin structure, in fact, induces the Spin structure on Y^{11} via $\pi : Y^{11} \rightarrow Y^{11}/S^1$; any Spin structure on X^{10} canonically induces a projectable Spin structure on Y^{11} via pullback. Let $\sigma_X : P_{\text{Spin}}(X^{10}) \rightarrow P_{SO}(X^{10})$ be a Spin structure on X^{10} , and let $\rho_{10} : \text{Spin}(10) \rightarrow SO(10)$ is the two-fold covering map. Then

$$\pi^* \sigma_X : \pi^* P_{\text{Spin}}(X^{10}) \rightarrow \pi^* P_{SO}(X^{10}) := P_{SO(10)}(Y^{11}) \quad (2.28)$$

is a ρ -equivariant map. A Spin structure on Y^{11} is now obtained by enlarging the structure group from Spin(10) to Spin(11)

$$\sigma_Y := \pi^* \sigma_X \times_{\rho_{10}} \rho_{11} : \pi^* P_{\text{Spin}}(X^{10}) \times_{\text{Spin}(10)} \text{Spin}(11) \longrightarrow P_{SO(10)}(Y^{11}) \times_{SO(10)} SO(11) . \quad (2.29)$$

Thus we conclude that an even type free circle action leads to a Spin structure on the quotient ten-dimensional manifold which induces the original one on Y^{11} in eleven dimensions (cf. [33]).

We can associate to the S^1 bundle $Y^{11} \rightarrow X^{10}$ the complex line bundle $L := Y^{11} \times_{S^1} \mathbb{C}$ with the natural connection given by $i\omega$. Consider the twisted Spin bundle $SX^{10} \otimes L^{-k}$, where L^{-k} is the tensor power of k copies of the line bundle L^{-1} . Note that there is an equality of spinor modules $\Delta_{10} = \Delta_{11}$.

The dilatino. The S^1 -action on $P_{\text{Spin}}(Y^{11})$ induces an action of S^1 on the associated Spin vector bundle $SY^{11} = P_{\text{Spin}}(Y^{11}) \times_{\text{Spin}(11)} \Delta_{11}$ which we denote κ . The action $\kappa(e^{it})$ maps a spinor with base point y to one with base point $y \cdot e^{it}$. The Lie derivative of a smooth spinor ψ in the direction of the Killing field v is

$$\mathcal{L}_v(\psi)(y) = \frac{d}{dt} \Big|_{t=0} \kappa(e^{-it})(\psi(y \cdot e^{it})) . \quad (2.30)$$

Let e_i be an orthonormal basis for the tangent space. Any r -form F_r acts on a spinor ψ by

$$\begin{aligned} \gamma(F_r)\psi &= F_r(e_{i_1}, \dots, e_{i_r}) \gamma(e_{i_1}) \cdots \gamma(e_{i_r}) \psi \\ &= F_r(e_{i_1}, \dots, e_{i_r}) \gamma(e_{i_1} \wedge \cdots \wedge e_{i_r}) \psi . \end{aligned} \quad (2.31)$$

The Christoffel symbols involving one 11-dimensional component are proportional to $\frac{1}{4}e^\phi \gamma(dw)\Psi$, so that the difference between the covariant derivative and the Lie derivative $\mathcal{L}_v \Psi$, with respect to the Killing vector field v of the spinor Ψ is

$$\nabla_v \Psi - \mathcal{L}_v \Psi = \frac{1}{4} e^{2\phi} \gamma(dw) \Psi . \quad (2.32)$$

Fourier modes. With A a spinor corresponding to the principal Spin bundle and σ_X a Spin structure, let

1. $[A, \sigma_X]$ denote the equivalence class of (A, σ_X) in $SX^{10} = P_{\text{Spin}}(X^{10}) \times_{\text{Spin}(10)} \Delta_{10}$.
2. $[\pi^* A, \sigma_X]$ denote the equivalence class of $(\pi^* A, \sigma_X)$ in $SY^{11} = \pi^* P_{\text{Spin}}(X^{10}) \times_{\text{Spin}(10)} \Delta_{10}$.
3. $[y, 1]$ denote the equivalence class of $(y, 1)$ in $L = Y^{11} \times_{S^1} \mathbb{C}$.

Define a vector bundle map

$$\begin{aligned} \Pi_k : SY^{11} &\longrightarrow SX^{10} \otimes L^{-k} \\ (y, [\pi^* A, \sigma_X]) &\longmapsto (\pi(y), [A, \sigma_X] \otimes [y, 1]^{-k}) . \end{aligned} \quad (2.33)$$

This is a fiberwise vector space isomorphism preserving Clifford multiplication. Therefore, for any such ten-dimensional spinor coupled to powers of the M-theory line bundle $\psi : X^{10} \rightarrow SX^{10} \otimes L^{-k}$ we get an eleven-dimensional spinor $\Psi = Q_k(\psi)$, with a homomorphism of Hilbert spaces $Q_k : L^2(SX^{10} \otimes L^{-k}) \rightarrow L^2(SY^{11})$ (injective since π is surjective), such that we have a commutative diagram

$$\begin{array}{ccc} Y^{11} & \xrightarrow{Q_k(\psi)} & SY^{11} \\ \downarrow \pi & & \downarrow \Pi_k \\ X^{10} & \xrightarrow{\psi} & SX^{10} \otimes L^{-k} . \end{array} \quad (2.34)$$

For any section $\Psi \in \Gamma(SY^{11})$, as in [3],

$$\Psi \in \text{Im}(Q_k) \iff \mathcal{L}_v \Psi = ik\Psi . \quad (2.35)$$

Since \mathcal{L}_v is the differential of a representation of the Lie group $S^1 = U(1)$ on $L^2(SY^{11})$, we get a decomposition $L^2(SY^{11}) = \bigoplus_{k \in \mathbb{Z}} V_k$ into eigenspaces V_k of \mathcal{L}_v for the eigenvalue ik , $k \in \mathbb{Z}$. This decomposition is preserved by the Dirac operator on Y^{11} because the S^1 -action commutes with that operator. So the image of Q_k is precisely V_k and the map Q_k is an isometry up to the factor e^ϕ . In fact, using [3], there is a homothety of Hilbert spaces $Q_k : L^2(SX^{10} \otimes L^{-k}) \rightarrow V_k$ such that the horizontal covariant derivative is given by

$$\nabla_{\tilde{X}} Q_k(\Psi) = Q_k(\nabla_X \psi) + \frac{1}{4} e^{2\phi} \gamma(e^{-\phi} v) \gamma(\tilde{V}_X) Q_k(\Psi) , \quad (2.36)$$

where \tilde{V}_X is the vector field on X^{10} satisfying $d\omega(\tilde{X}, \cdot) = \langle \tilde{V}_X, \cdot \rangle$, and such that Clifford multiplication is preserved

$$Q_k(\gamma(X)\psi) = \gamma(\tilde{X})Q_k(\Psi) . \quad (2.37)$$

This can be viewed as dimensional reduction.

Relating spectra. Let D be the Dirac operator on SX^{10} . Let $E \rightarrow X^{10}$ be a Hermitian vector bundle with a metric connection ∇^E and let D^E be the twisted Dirac operator on $SX^{10} \otimes E$. This splits into three parts (see [3]).

1. The *horizontal Dirac operator* is defined as the unique closed linear operator $D_h : L^2(SY^{11}) \rightarrow L^2(SY^{11})$ on each V_k given by $D_h := Q_k \circ D \circ Q_k^{-1}$, where D is the (twisted) Dirac operator on $SX^{10} \otimes L^{-k}$.
2. The *vertical Dirac operator* is defined to be $D_v := \gamma(e^{-\phi} v) \gamma(d\omega)$.
3. The *zeroth order term* $D_0 := -\frac{1}{4} \gamma(e^{-\phi} v) \gamma(d\omega)$.

The Dirac operator \tilde{D}^ϕ decomposes as

$$\tilde{D}^\phi = \sum_{i=0}^{10} \gamma(e_i) \nabla_{e_i} = e^{-\phi} D_v + D_h + e^\phi D_0 . \quad (2.38)$$

Let μ_1, μ_2, \dots be the eigenvalues of D^E . The eigenvalues of the twisted Dirac operator \tilde{D}^ϕ for \tilde{g}_Y^ϕ on $SY^{11} \otimes \pi^* E \rightarrow Y^{11}$ depend continuously on e^ϕ and such that for $e^\phi \rightarrow 0$ [3]:

1. For any $j \in \mathbb{N}$ and $k \in \mathbb{Z}$, $e^\phi \cdot \lambda_{j,k}(e^\phi) \rightarrow k$. In particular, $\lambda_{j,k}(e^\phi) \rightarrow \pm\infty$ if $k \neq 0$.
2. $\lambda_{j,0}(e^\phi) \rightarrow \mu_j$. The convergence of the eigenvalues $\lambda_{j,0}(e^\phi)$ is uniform in j .

We interpret this physically as the behavior of the eigenvalues in the weak coupling limit, since we view e^ϕ as essentially the string coupling parameter.

Example: Berger sphere. Consider the Hopf fibration $S^{11} \rightarrow \mathbb{CP}^5$. The Spin structure on S^{11} is projectable since \mathbb{CP}^5 is Spin. The Dirac operator of S^{11} with Berger metric includes the eigenvalues $\lambda(\phi) = e^{-\phi}(m+3) + \frac{5}{2}e^{\phi}$ with $m \in \mathbb{N}_0$, with multiplicity $\binom{m+5}{m}$. Note that $\lim_{e^{\phi} \rightarrow 0} e^{\phi} \cdot \lambda(\phi) = \pm(m+3)$. For the complete set of eigenvalues see [3].

The set of Spin structures on Y^{11} . If Y^{11} is an eleven-dimensional Spin manifold then $w_2(Y^{11}) = 0$. The Gysin sequence of π gives that $w_2(X^{10}) = 0$ or $w_2(X^{10}) = w_2(\pi) = c_1(\pi) \bmod 2$. Note that, from the point of view of modes on the circle, this corresponds to even vs. odd modes, i.e. the Kaluza-Klein level k being even or odd.

1. The Spin structure σ_Y on Y^{11} is *equivariant* if and only if X is a Spin manifold with Spin structure σ_X and $\sigma_Y = \pi^* \sigma_X$. Such a Spin structure does not extend to a Spin structure on the associated disk bundle Z^{12} . The induced Spin^c -Dirac structure ¹ is *strictly equivariant*.
2. If σ_Y is not equivariant then there is a Spin structure σ_Z on Z^{12} with $\sigma_Y = \partial \sigma_Z$: σ_Y induces a Spin structure on $\pi^* P_{SO}(X^{10})$.

The set $\text{Spin}(Y^{11})$ of isomorphism classes of Spin structures on Y^{11} , when X and π are both Spin, is given by: $\text{Spin}(Y^{11}) = \pi^* \text{Spin}(X^{10}) \cup \partial \text{Spin}(Z^{12})$.

2.6 Nonprojectable Spin structures

Now assume that the circle action χ is of odd type. Here Spin^c structures make an appearance (see section 3.1 for basic definitions and examples). We will continue to make use of the construction in [3]. Let $\sigma_Y : P_{\text{Spin}}(Y^{11}) \rightarrow P_{SO}(Y^{11})$ be a nonprojectable Spin structure on Y^{11} . In this case, X^{10} is not necessarily Spin. We can form a principal $\text{Spin}(10)$ bundle $P := \sigma_Y^{-1}(P_{SO(10)}(Y^{11}))$ by restricting the principal $SO(11)$ bundle $P_{SO(11)}(Y^{11})$ to a principal $SO(10)$ bundle $P_{SO(10)}(Y^{11})$. The action of $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ does not lift to P , but the double covering of S^1 , i.e. $S^1 \cong \mathbb{R}/4\pi\mathbb{Z}$ acts on P . A free $\text{Spin}^c(11) = \text{Spin}(11) \times_{\mathbb{Z}_2} \hat{S}^1$ action on $P_{\text{Spin}}(Y^{11})$ is induced from the free action $\hat{\chi} : P_{\text{Spin}}(Y^{11}) \times (\text{Spin}(11) \times \hat{S}^1) \rightarrow P_{\text{Spin}}(Y^{11})$, defined by $\hat{\chi}(\psi, (g, \lambda)) = \lambda^{-1} \psi g$, where $\psi \in P_{\text{Spin}}(Y^{11})$, $g \in \text{Spin}(11)$ and $\lambda \in \hat{S}^1$. A $\text{Spin}^c(11)$ bundle over Y^{11}/S^1 is the composition $P_{\text{Spin}}(Y^{11}) \rightarrow Y^{11} \rightarrow Y^{11}/S^1$. Now we obtain a Spin^c structure on $\text{Fr}(Y^{11})/S^1$ and on $T(Y^{11}/S^1)$ as follows. Define the map $\tau : P_{\text{Spin}}(Y^{11}) \rightarrow \text{Fr}(Y^{11})/S^1 \times Y^{11}$ by $\tau(\psi) = (\pi_{\text{Fr}}(u(\psi)), p(\psi))$, which is equivariant so that the map $P_{\text{Spin}}(Y^{11}) \rightarrow Y^{11}/S^1$ is the composite map $\rho_F \tau$, where $\rho_F : \text{Fr}(Y^{11})/S^1 \times Y^{11} \rightarrow Y^{11}/S^1$. Thus, for a circle action of odd type, a Spin structure on Y^{11} gives a Spin^c structure on the quotient ten-dimensional manifold Y^{11}/S^1 , with π is a principal $U(1)$ bundle, such that the principal $\text{Spin}^c(11)$ -bundle is given by the composition of the principal $\text{Spin}(11)$ -bundle over Y^{11} and π .

Define $\text{Spin}^c(10) = \text{Spin}(10) \times_{\mathbb{Z}_2} S^1$ where $-1 \in \mathbb{Z}_2$ identifies $(-A, c)$ with $(A, -c)$. The Lie group $\text{Spin}^c(10)$ acts on Δ_{10} . The actions of $\text{Spin}(10)$ and S^1 on P induce a free action of $\text{Spin}^c(10)$ on P , so that P can be viewed as a principal $\text{Spin}^c(10)$ bundle over X^{10} . The associated vector bundle is $P \times_{\text{Spin}^c(10)} \Delta_{10}$, which is $SX^{10} \otimes L^{1/2}$ when X^{10} is Spin. When X^{10} is not Spin, then neither factor exist, but the product $SX^{10} \otimes L^{1/2}$ exists. See section 4.2 for more on how the M-theory circle is used to construct a Spin^c structure. This is one of the main motivations for considering Spin^c structures in ten dimensions.

For the current case of nonprojectable Spin structure on Y^{11} , we get a splitting

$$L^2(SY^{11}) = \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} V_k \quad (2.39)$$

into eigenspaces V_k for \mathcal{L}_v corresponding to the eigenvalue ik . Therefore, taking the base ten-dimensional manifold X^{10} to be Spin^c , while Y^{11} is Spin, leads to half-integral Fourier modes for the spinors on Y^{11} . This seems to be a new phenomenon in the Fourier decomposition of spinor modes in M-theory reduction on the circle and hence complements the discussion in [54].

¹A Spin structure induces a Spin^c -structure (cf. section 3).

Relating spectra. Let the Spin structure on Y^{11} be nonprojectable. The eigenvalues $(\lambda_{j,k}(\phi))_{j \in \mathbb{N}, k \in \mathbb{Z} + \frac{1}{2}}$ of the twisted Dirac operator D_Y^ϕ depend continuously on ϕ and for $e^\phi \rightarrow 0$

$$e^\phi \cdot \lambda_{j,k}(\phi) \rightarrow k \quad \text{for all } j = 1, 2, \dots \quad (2.40)$$

In particular, $\lambda_{j,k}(\phi) \rightarrow \pm\infty$. As in the case of projectable Spin structures in the last section, we interpret this physically as the behavior of the eigenvalues in the weak coupling limit.

Spin structures on Y^{11} from Spin^c structures on X^{10} . Since $U(1) = SO(2)$, there is a natural map $SO(10) \times U(1) \rightarrow SO(12)$ which extends, via Whitney sum, to a map of bundles. Then we can define $P_{\text{Spin}^c}(X^{10})$ as the pullback by this map of this covering map

$$\begin{array}{ccc} P_{\text{Spin}^c}(X^{10}) & \xrightarrow{\quad} & P_{\text{Spin}}(L) \\ \downarrow & & \downarrow \\ SO(10) \times U(1) & \xrightarrow{\quad} & SO(12) \end{array} \quad (2.41)$$

Therefore, a Spin^c structure on TX^{10} consists of a complex line bundle L and a Spin structure on $TX^{10} \oplus L$. In fact, every Spin^c structure on X^{10} induces a canonical Spin structure on Y^{11} . From [127], by enlargement of the structure groups, the two-fold covering $\theta : P_{\text{Spin}^c(10)}(X^{10}) \rightarrow P_{SO(10)}X^{10} \times P_{U(1)}(X^{10})$ gives a two-fold covering $\theta : P_{\text{Spin}^c(11)}(X^{10}) \rightarrow P_{SO(11)}(X^{10})$ which, by pullback through π gives rise to a Spin^c structure on Y^{11}

$$\begin{array}{ccc} P_{\text{Spin}^c(11)}(Y^{11}) & \xrightarrow{\quad \pi \quad} & P_{\text{Spin}^c(11)}(X^{10}) \\ \downarrow \pi^* \theta & & \downarrow \theta \\ P_{SO(11)}(Y^{11}) \times P_{U(1)}(Y^{11}) & & P_{SO(11)}(X^{10}) \times P_{U(1)}(X^{10}) \\ \downarrow & & \downarrow \\ Y^{11} & \xrightarrow{\quad \pi \quad} & X^{10} \end{array} \quad (2.42)$$

Since the pullback of a principal G -bundle with respect to its own projection map is always trivial then this gives a Spin structure on Y^{11} .

2.7 Dependence of the phase on the Spin structure

Consider the principal $U(1)$ bundle $\pi : Y^{11} \rightarrow X^{10}$. Fixing a basis vector of the Lie algebra $\mathfrak{u}(1)$ trivializes the vertical tangent bundle $T^v\pi$. Choose a Spin structure on $T^v\pi$ which restricts to the trivial (i.e. bounding) Spin structure on each fiber. Since $T^v\pi$ is trivial, The Gysin sequence for the circle bundle gives

$$0 \longrightarrow H^1(X^{10}; \mathbb{Z}) \longrightarrow H^1(Y^{11}; \mathbb{Z}) \xrightarrow{\text{res}} H^0(X^{10}; \mathbb{Z}) \xrightarrow{d_2} H^2(X^{10}; \mathbb{Z}) \quad (2.43)$$

The map $\text{res} : H^1(Y^{11}; \mathbb{Z}) \rightarrow H^0(X^{10}; \mathbb{Z}) \cong \mathbb{Z} \cong H^1(S^1; \mathbb{Z})$ is the restriction to the fiber. Acting on the generator $1 \in H^0(X^{10}; \mathbb{Z}) \cong \mathbb{Z}$, d_2 gives $d_2(1) = -c_1(Y^{11})$, so that the reduction modulo two of (2.43) gives

$$0 \longrightarrow H^1(X^{10}; \mathbb{Z}_2) \longrightarrow H^1(Y^{11}; \mathbb{Z}_2) \xrightarrow{r_2(\text{res})} H^0(X^{10}; \mathbb{Z}_2) \xrightarrow{r_2(c_1(Y^{11}))} H^2(X^{10}; \mathbb{Z}_2) \quad (2.44)$$

A Spin structure of $T^v\pi$ corresponding to $x \in H^1(Y^{11}; \mathbb{Z}_2)$ restricts to the trivial Spin structure on the fibers iff the mod 2 reduction $r_2(\text{res})(x) \neq 0$. Since $H^0(X^{10}; \mathbb{Z}_2) = \mathbb{Z}_2$, the condition is $r_2(c_1(Y^{11})) = 0$. Therefore, the vertical tangent bundle $T^v\pi$ admits a Spin structure which restricts to the trivial Spin structure on the fibers iff the reduction modulo 2 of the Euler class $c_1(Y^{11})$ vanishes.

The dependence of the (exponentiated) eta invariant on the Spin structure. Let σ_Y be a Spin structure on Y^{11} . We will study the dependence of the phase of the M-theory partition function, via the eta invariant, on the Spin structure. To indicate which Spin structure is being considered, we will use a label for the Spin bundle, e.g. S_{σ_Y} to denote the Spin bundle over Y^{11} corresponding to σ_Y . The eta invariant for the twisted Dirac operator D^E acting on sections of $S_{\sigma_Y} \otimes E$ with a given Spin structure σ_Y is $\eta(\sigma_Y; E)$. As we will see, E for us will be either an E_8 bundle or $TY^{11} - 3\mathcal{O}$. Since the set of Spin structures $\text{Spin}(Y^{11})$ on Y^{11} is a torsor over $H^1(Y^{11}; \mathbb{Z}_2)$, a new Spin structure σ'_Y is obtained from an old one σ_Y by acting with $H^1(Y^{11}; \mathbb{Z})$ on that set:

$$\begin{aligned} \text{Spin}(Y^{11}) \times H^1(Y^{11}; \mathbb{Z}_2) &\longrightarrow \text{Spin}(Y^{11}) \\ (\sigma_Y, \delta) &\longmapsto \sigma_Y + \delta. \end{aligned} \quad (2.45)$$

We would like to compare the eta invariant corresponding to σ_Y , $\eta(\sigma_Y; E)$, to that corresponding to σ'_Y , $\eta(\sigma_Y + \delta; E)$, i.e.

$$\Delta\eta(\sigma_Y, \delta; E) = \eta(\sigma_Y + \delta; E) - \eta(\sigma_Y; E), \quad (2.46)$$

which is a special case of Atiyah-Patodi-Singer (APS) relative eta invariant [12].

The effect of δ will be seen via a line bundle, whose curvature will essentially appear in the APS index formula. To illustrate our point it is enough to consider the case when the complex line bundle L_δ is topologically trivial, and hence admits a section. The section ℓ defines $\beta = \frac{1}{\pi i} \ell^{-1} d\ell$, which is a closed real-valued one-form on Y^{11} . The Dirac operator D' twisted by ℓ is related to D via

$$D' = \ell^{-1} D \ell = D + \ell^{-1} \text{grad } \ell. \quad (2.47)$$

Both D and D' are twisted by E to give D^E and D'^E , respectively. It is shown in [46] that D' acting on $\Gamma(S_{\sigma_Y} \otimes E)$ has the same spectrum as D acting on $\Gamma(S_{\sigma'_Y} \otimes E)$, so that the difference of their η -invariants

$$\eta_{D'^E} - \eta_{D^E} = \eta(\sigma_Y + \delta; E) - \eta(\sigma_Y; E) = \Delta\eta(\sigma_Y, \delta; E), \quad (2.48)$$

the difference between the η -invariants corresponding to different Spin structures. The η -invariant part of the APS index theorem in this case will involve exactly the above combination, as we see in what follows.

Let $Z^{12} = Y^{11} \times [0, 1]$. The Spin structure on Y^{11} induces a Spin structure on Z^{12} with associated Spin bundle $S^Z := S_{\sigma_Y} \oplus S_{\sigma_Y}$. The Spin structure line bundle L_δ gives rise to a trivial complex line bundle L_{tr} with connection ∇^L over Z^{12} . The main goal will be to identify the contribution to the index from this line bundle. Near the boundary components of Z^{12} , the corresponding Dirac operator \overline{D}^E acting on sections of $S^Z \otimes E \otimes L_{tr}$ will have the form

$$\begin{aligned} \overline{D}^E &= \frac{\partial}{\partial t} + D^E \quad \text{near } Y^{11} \times \{0\}, \\ \overline{D}^E &= \frac{\partial}{\partial t} + D'^E \quad \text{near } Y^{11} \times \{1\}. \end{aligned} \quad (2.49)$$

The Atiyah-Patodi-Singer index theorem in this case is [46]

$$\text{Ind}(\overline{D}^E) = \int_{Z^{12}} \widehat{A}(Y^{11}) \wedge \text{ch}(\nabla^E) \wedge \text{ch}(\nabla^L) - \eta_{D^E} + \eta_{D'^E}. \quad (2.50)$$

The eta factors, by the fact that D'^E acting on $\Gamma(S_{\sigma_Y} \otimes E)$ is isospectral to D^E acting on $\Gamma(S_{\sigma_Y + \delta} \otimes E)$, are the differences (2.46), so that

$$\Delta\eta(\sigma_Y, \delta; E) = -\text{Ind}(\overline{D}^E) + \int_{Z^{12}} \widehat{A}(Y^{11}) \wedge \text{ch}(\nabla^E) \wedge \text{ch}(\nabla^L). \quad (2.51)$$

Now we relate the integral, involving a polynomial I_{4m+2} in characteristic classes, over the manifold with boundary Z^{12} to an integral over the manifold with no boundary $Y^{11} \times S^1$ as

$$\int_{Z^{12}} I_{4m+2} \wedge \text{ch}(\nabla^L) = -\frac{1}{2} \int_{Y^{11}} I_{4m+2} \wedge \beta = -\frac{1}{2} \int_{Y^{11} \times S^1} I_{4m+2} \wedge e^{dt \wedge \beta}. \quad (2.52)$$

For this we need two smooth scalar functions. The first is $\chi : I \rightarrow I$ given by $\chi(t) = 0$ for $t \leq 1/3$ and $\chi(t) = 1$ for $t \geq 2/3$, with $\text{ch}_1(\nabla^L) = -\frac{1}{2}d\chi \wedge \beta$. The second is dt , which measures the size of the circle. Since $dt \wedge \beta \in H^2(Y^{11} \times S^1; \mathbb{Z})$, the exponential in (2.52) can be viewed as the Chern character of a complex line bundle L_S on $Y^{11} \times S^1$. Then the integral in (2.50) is $-\frac{1}{2}\text{Ind}(D^{E \otimes L_S})$. Therefore, from (2.51), we get

$$\Delta\eta(\sigma_Y, \delta; E) = -\text{Ind}(\overline{D}^E) - \frac{1}{2}\text{Ind}(D^{E \otimes L_S}) \in \frac{1}{2}\mathbb{Z}. \quad (2.53)$$

The above analysis for the E_8 bundle can be repeated for the Rarita-Schwinger bundle $RS = TY^{11} - 3\mathcal{O}$, giving

$$\Delta\eta(\sigma_Y, \delta; RS) = -\text{Ind}(\overline{D}^{RS}) - \frac{1}{2}\text{Ind}(D^{RS \otimes L_S}) \in \frac{1}{2}\mathbb{Z}. \quad (2.54)$$

We are now ready to look at the effect on the partition function. Using the notation in this section and restoring the Spin structure labels, we have that the phase (1.1) of the partition function varies with the change in Spin structure as

$$\begin{aligned} \Phi_{\sigma_Y + \delta} &= \Phi_{\sigma_Y} \exp \left[2\pi i \left(\frac{1}{2} \Delta\overline{\eta}(\sigma_Y + \delta; E) + \frac{1}{4} \Delta\overline{\eta}(\sigma_Y + \delta; RS) \right) \right] \\ &= \Phi_{\sigma_Y} \exp \left[2\pi i \left(\frac{1}{4} (\Delta h_{E_8} + \Delta\eta(\sigma_Y, \delta; E)) + \frac{1}{8} (\Delta h_{RS} + \Delta\eta(\sigma_Y, \delta; RS)) \right) \right], \end{aligned} \quad (2.55)$$

We have two variations, one from the zero modes Δh and one from the (unreduced) η -invariants. We have seen in section 2.3 that the spectrum and the number of zero modes of the Dirac operator depend crucially on the Spin structure chosen. Hence, we do not expect Δh to be zero, nor to be a multiple of 4 or 8. On the other hand, we have seen above in (2.54) that the variation of the η -invariant $\Delta\eta$ is in general only half-integral and does not necessarily have divisors. Hence, in general, the contribution to the phase from the variation of the Spin structure is not unity and therefore we view such possible multi-valuedness as an anomaly. We call this anomaly a *Spin structure anomaly*.

Special case: flat connections. Let us consider the case when the bundle, say E , admits a flat connection ∇^E . In this case,

$$\begin{aligned} \Delta\eta(\sigma_Y, \eta; E) &= -\text{Ind}(\overline{D}^E) - \frac{1}{2} \int_{Y^{11}} \widehat{A}(Y^{11}) \wedge \text{ch}(\nabla^E) \wedge \beta \\ &= -\text{Ind}(\overline{D}^E) - \frac{1}{2} \int_{Y^{11}} \widehat{A}(Y^{11}) \wedge \beta \\ &= -\text{Ind}(\overline{D}^E) \in \mathbb{Z}, \end{aligned} \quad (2.56)$$

where we have used the fact that the integrand in the middle line is of degree $4m+1$, $m \geq 0$, which cannot match 11, thus giving value zero for the integral. The situation in the flat case is a bit better than the general case. However, it is not enough to ensure the absence of potential anomalies in all such situations. In dimension twelve the \widehat{A} -genus of a Spin manifold is even (see section 5.2) so at least for trivial E the index will be in $2\mathbb{Z}$, but that would still lead to a discrepancy in the phase of $1/2$ and $\frac{1}{4}$ for E and $TY^{11} - \mathcal{O}$, respectively. We will need further divisibility of the index. While this is not true in general, it is certainly not unattainable in special cases.

Example: Flat manifolds with holonomy groups of finite prime order. Consider flat manifolds Y^{11} whose holonomy groups have prime order. Let e_1, \dots, e_{11} be a basis of \mathbb{R}^{11} . Consider the map $J(x) = A(x) + \frac{1}{11}e_{11}$, with $A : \mathbb{R}^{11} \rightarrow \mathbb{R}^{11}$ given by

$$A(e_j) = \begin{cases} e_{j+1} & \text{for } j \leq 9, \\ -(e_1 + \dots + e_{10}) & \text{for } j = 10, \\ e_{11} & \text{for } j = 11. \end{cases} \quad (2.57)$$

Then $Y^{11} = \mathbb{R}^{11}/\Gamma$ with $\Gamma = \langle e_1, \dots, e_{10}, J \rangle$. The linear part A of J has two lifts $s_{\pm} \in \text{Spin}(11)$ such that $(s_+)^{11} = \text{id}$ and $(s_-)^{11} = -\text{id}$. This defines two Spin structures σ_{\pm} on Y^{11} . For every $\epsilon = (\epsilon_1, \dots, \epsilon_5) \in \mathbb{Z}_2^5$, consider $\mu_{\epsilon} = \sum_{j=1}^5 j\epsilon_j$ and consider elements of product 1, i.e. elements such that $\nu = \epsilon_1 \dots \epsilon_5 = 1$, and let $\mathcal{C}_+ = \{\epsilon \in \mathbb{Z}_2^5 : \nu(\epsilon) = 1\}$. Following [147], the values of the eta invariants corresponding to s_+ and s_- , respectively are

$$\eta_{Y^{11}, \sigma_+} = \sum_{r=1}^{10} A_r^+ \left(1 - \frac{2r}{11}\right), \quad \eta_{Y^{11}, \sigma_-} = \sum_{r=0}^{10} A_r^- \left(1 - \frac{2r+1}{11}\right), \quad (2.58)$$

where $A_r^{\pm} \in 2\mathbb{Z}$ is twice the number of elements in \mathcal{C}_+ which satisfy

$$\frac{\mu_{\epsilon} + 11}{2} \equiv r \pmod{11}, \quad \frac{\mu_{\epsilon} + 21}{2} \equiv r \pmod{11}, \quad (2.59)$$

for σ_+ and σ_- , respectively, for $r \in \{1, 2, \dots, 10\}$. The above eta invariants (2.57) are integral [147]. This can be seen as follows. From [156], 2^s copies of Y^{11} , for some $s \in \mathbb{N}$, is a boundary of a Spin twelve-manifold Z^{12} . From the APS index theorem [11] $\int_{Z^{12}} \hat{A}(Z^{12}) - \frac{1}{2}2^s \eta_{Y^{11}} \in \mathbb{Z}$. From [92], the integral can be written as $\frac{1}{q_1 \dots q_r} C_{Z^{12}}$, where $C_{Z^{12}} \in \mathbb{Z}$ and $q_1, \dots, q_r \in \{2, 3, \dots, 10\}$ are primes numbers. Then from (2.57), $\eta_{Y^{11}} = \frac{1}{11} C_{Y^{11}}$ for some $C_{Y^{11}} \in \mathbb{Z}$. Since $\frac{1}{q_1 \dots q_r} C_{Z^{12}} - 2^{s-1} \eta_{Y^{11}} \in \mathbb{Z}$ then $\frac{2^{s-1} q_1 \dots q_r}{11} C_{Y^{11}} \in \mathbb{Z}$, which implies that $\eta_{Y^{11}} \in \mathbb{Z}$. Now the difference of eta invariants $\Delta \eta_{Y^{11}} = \eta_{Y^{11}, \sigma_+} - \eta_{Y^{11}, \sigma_-}$ takes values in $\frac{1}{2}\mathbb{Z}$, by (2.53). Now $A_r^{\pm} \in 2\mathbb{Z}$ so that $\Delta \eta_{Y^{11}} = (\frac{2}{c})(11)$ for some $c \in \mathbb{Z}$. Summing up, gives $\frac{11}{2} \Delta \eta_{Y^{11}} \in \mathbb{Z}$, so that $\Delta \eta_{Y^{11}} \in 2\mathbb{Z}$.

The dimension of the space of harmonic spinors can also be obtained from the above data [147]. For $\dim M = 2k + 1$,

$$\begin{aligned} h(M, \sigma^+) &= 2\# \left\{ \epsilon \in \mathcal{C}_+ \mid \frac{\mu_{\epsilon}}{2} + c(k)n \equiv 0 \pmod{2k+1} \right\}, \\ h(M, \sigma^-) &= 0. \end{aligned} \quad (2.60)$$

We work out examples in three, seven and eleven dimensions. The first two (already considered in [147]) are relevant for considering them as fibers of bundles of dimension eleven. We work out the latter case, which is the most important for our discussion.

1. Three dimensions: In the case of a Spin three-manifold M^3 , we have μ_{ϵ} and $\nu(\epsilon)$ both equal to 1 for the only element ϵ . From $\frac{\mu_{\epsilon}}{2} + \frac{1}{2} \dim M^3 = 2$ we get that the only nonzero component of A_r^+ is $A_2^+ = 2$. Then,

$$\eta_{M^3, \sigma^+} = \sum_{r=1}^2 A_r^+ \left(1 - \frac{2r}{3}\right) = -\frac{2}{3}. \quad (2.61)$$

On the other hand, from $\frac{\mu_{\epsilon}}{2} + \frac{1}{2} \dim M^3 + 1 = 3$ we get that the only nonzero component of A_r^- is $A_0^- = 2$. Then

$$\eta_{M^3, \sigma^-} = \sum_{r=0}^2 A_r^- \left(1 - \frac{2r+1}{3}\right) = \frac{4}{3}. \quad (2.62)$$

Note that here that η_{M^3, σ^\pm} are not integral. However, the difference $\Delta\eta_{M^3} = \eta_{M^3, \sigma^+} - \eta_{M^3, \sigma^-} = -2 \in 2\mathbb{Z}$. Here there are no harmonic spinors, i.e. $h(M^3, \sigma^\pm) = 0$.

2. *Seven dimensions:* In order to consider elements in \mathcal{C}_+ , we need an even number of -1 's in ϵ . For σ^+ we have the relation $r \equiv \frac{\mu_\epsilon}{2} \pmod{7}$, while for σ^- we have $r' \equiv \frac{\mu_\epsilon}{2} + 3 \pmod{7}$. The values of the parameters are summarized in this table

ϵ	$\frac{\mu_\epsilon}{2}$	r	$\frac{\mu_\epsilon}{2} + 3$	r'
(1, 1, 1)	3	3	6	6
(1, -1, -1)	-2	5	1	1
(-1, 1, -1)	-1	6	2	2
(-1, -1, 1)	0	0	3	3

(2.63)

We see that $A_j^+ = 2$ for $j = 0, 3, 5, 6$ and $A_j^+ = 0$ for all other values of j . Also, $A_j^- = 2$ for $j = 1, 2, 3, 6$ and $A_j^- = 0$ for all other values of j . From this we get

$$\eta_{M^7, \sigma^+} = -2, \quad \eta_{M^7, \sigma^-} = 0. \quad (2.64)$$

This gives $\Delta\eta_{M^7} = \eta_{M^7, \sigma^+} - \eta_{M^7, \sigma^-} = -2 \in 2\mathbb{Z}$. The last row in the above table gives that the dimension of the space of harmonic spinors corresponding to σ^+ is $h(M^7, \sigma^+) = 2 \cdot 1 = 2$.

3. *Eleven dimensions:* Again, in order to get elements in \mathcal{C}_+ , we need ϵ 's with an even number (i.e. 0, 2, or 4) of minuses. The parameters then are encoded in this table

ϵ	$\frac{\mu_\epsilon}{2}$	$\frac{\mu_\epsilon + 11}{2}$	$\frac{\mu_\epsilon + 11}{2} \pmod{11}$	$\frac{\mu_\epsilon + 21}{2}$	$\frac{\mu_\epsilon + 21}{2} \pmod{11}$
(1, 1, 1, 1, 1)	$\frac{15}{2}$	13	2	18	7
(-1, -1, 1, 1, 1)	$\frac{9}{2}$	10	10	15	4
(-1, 1, -1, 1, 1)	$\frac{7}{2}$	9	9	14	3
(-1, 1, 1, -1, 1)	$\frac{5}{2}$	8	8	13	2
(-1, 1, 1, 1, -1)	$\frac{3}{2}$	7	7	12	1
(1, -1, -1, 1, 1)	$\frac{5}{2}$	8	8	13	2
(1, -1, 1, -1, 1)	$\frac{3}{2}$	7	7	12	1
(1, -1, 1, 1, -1)	$\frac{1}{2}$	6	6	11	0
(1, 1, -1, -1, 1)	$\frac{1}{2}$	4	4	9	9
(1, 1, -1, 1, -1)	$-\frac{1}{2}$	5	5	10	10
(1, 1, 1, -1, -1)	$-\frac{3}{2}$	4	4	9	9
(-1, -1, -1, -1, 1)	$-\frac{5}{2}$	3	3	8	8
(1, -1, -1, -1, -1)	$-\frac{13}{2}$	-1	10	4	4
(-1, 1, -1, -1, -1)	$-\frac{11}{2}$	0	0	5	5
(-1, -1, 1, -1, -1)	$-\frac{9}{2}$	1	1	6	6
(-1, -1, -1, 1, -1)	$-\frac{7}{2}$	2	2	7	7

This gives the following values for A_i^+ : 2, 2, 4, 2, 2, 2, 4, 4, 4, 2, 4 for $i = 0, 1, \dots, 10$. It also gives the following values for A_i^- : 4, 4, 4, 2, 4, 2, 2, 4, 2, 2, 2. Then a straightforward computation shows that the values of the eta invariants are

$$\eta_{Y^{11}, \sigma^+} = -2, \quad \eta_{Y^{11}, \sigma^-} = 4. \quad (2.65)$$

The difference between the two eta invariants is $\Delta\eta_{Y^{11}} = \eta_{Y^{11}, \sigma^+} - \eta_{Y^{11}, \sigma^-} = -6$, which is indeed an element of $2\mathbb{Z}$. Here the space of harmonic spinors corresponding to σ^+ can be read off from the third row from the bottom of the table, which gives $h(Y^{11}, \sigma^+) = 2 \cdot 1 = 2$.

More examples can be straightforwardly constructed using the properties of the η -invariant and the \hat{A} -genus under products and bundles (see section 5.2.1).

Exponentiated eta invariants. When dealing with phases and partition functions, it is more appropriate to use exponentiated eta invariants such as in [44]. This is applied in [71] for the case of the M-theory effective action, where it is called “setting the quantum integrand”. Since we are dealing with adiabatic limits of the eta invariants, and also with eta-forms, we found it more transparent to use the ‘unexponentiated’ form. Of course, in the end, we have exponentiated the results that we got in order to study the phase of the partition function. Therefore, our results hold at the level of exponentiated effective actions.

3 M-Theory on Eleven-Dimensional Spin^c Manifolds

3.1 Basic definitions and properties of Spin^c structures

We recall some facts about Spin^c structures, mainly following [9] [80] (Appendix D) and [107] (Appendix D). The Spin group $\text{Spin}(n)$ is the connected double cover of the orthogonal group $SO(n)$. Form the product $\text{Spin}(n) \times U(1)$. The group $\text{Spin}^c(n)$ is defined to be the quotient of the above product by the group \mathbb{Z}_2 composed of the elements $(\epsilon, -1)$, where ϵ is the non-trivial element of the kernel of the double covering map $\rho_{\text{spin}} : \text{Spin}(n) \rightarrow SO(n)$. This means that elements of $\text{Spin}^c(n)$ are equivalence classes $[s, \gamma]$, where $g \in \text{Spin}(n)$ and $\lambda \in U(1)$, so that $[g\epsilon, \lambda] = [g, -\lambda]$. There are two natural homomorphisms coming from the projection to the two factors, i.e. $\rho^c : \text{Spin}^c(n) \rightarrow SO(n)$ sending $[g, \lambda]$ to $\rho_{\text{spin}}(g)$ and $\det : \text{Spin}^c(n) \rightarrow U(1)$ taking $[g, \lambda]$ to λ^2 which give rise to the short exact sequence

$$1 \longrightarrow U(1) \longrightarrow \text{Spin}^c(n) \xrightarrow{\rho^c} SO(n) \longrightarrow 1, \quad (3.1)$$

$$1 \longrightarrow \text{Spin}(n) \longrightarrow \text{Spin}^c(n) \xrightarrow{\det} U(1) \longrightarrow 1. \quad (3.2)$$

Let M be a Riemannian manifold and ζ a real oriented vector bundle over M of rank n with oriented orthonormal frame bundle $\zeta : P_{SO}(\zeta) \rightarrow M$. Let $\rho : \text{Spin}(n) \rightarrow SO(n)$ be the nontrivial double covering, as above, and define the induced representation $\rho^c : \text{Spin}^c(n) := \text{Spin}(n) \times_{\mathbb{Z}_2} U(1) \rightarrow SO(n)$, which is a nontrivial principal $U(1)$ bundle over $SO(n)$. Thus a Spin^c structure on ζ is a principal $U(1)$ bundle $\sigma^c : P_{\text{Spin}^c}(\zeta) \rightarrow P_{SO}(\zeta)$, whose restriction to any fiber of $P_{SO}(\zeta)$ is the canonical principal $U(1)$ bundle ρ^c . This σ^c is called a Spin^c -structure on X^{10} . The canonical $U(1)$ bundle of σ^c is $\xi(\sigma^c) : P_{U(1)}(\sigma^c) := P_{\text{Spin}^c}(\zeta) \times_{\text{Spin}^c(n)} U(1) \rightarrow M$, so that the following diagram is commutative

$$\begin{array}{ccccc} P_{SO}(\zeta) & \xleftarrow{\tilde{\xi}} & P_{SO}(\zeta) \times_M P_{U(1)}(\sigma^c) & \xleftarrow{\sigma^{c'}} & P_{\text{Spin}^c}(\zeta) \\ \zeta \downarrow & & \downarrow & & \\ M & \xleftarrow{\xi(\sigma^c)} & P_{U(1)}(\sigma^c) & & \end{array} \quad (3.3)$$

where $\sigma^{c'}$ is a twofold covering, $\sigma^c = \tilde{\xi} \circ \sigma^{c'}$, and the square is a pullback diagram. When $\zeta = TM$ then we talk about Spin^c structures on the manifold M and write $P_{SO}(M)$. When M has a boundary then on $N = \partial M$ the boundary Spin^c -structure $\partial\sigma^c$ is the restriction of σ^c to $P_{SO}(N) \hookrightarrow P_{SO}(M)$. Note that the reduction modulo 2 of the first Chern class $c_1(\sigma^c) := c_1(\xi(\sigma^c)) \in H^2(M; \mathbb{Z})$ is the second Stiefel-Whitney class $w_2(\zeta)$ and Spin^c structures on ζ exist if $w_2(\zeta)$ is the reduction modulo 2 of an integral class $c \in H^2(M; \mathbb{Z})$.

Following [22] [23], a Spin^c -Dirac structure $(\sigma^c, \omega^{\sigma^c})$ on M is a Spin^c -structure σ^c together with a connection on $P_{\text{Spin}^c}(M) \rightarrow M$ which is compatible with the Levi-Civita connection.

The line bundles. There are two line bundles corresponding to the homomorphisms (3.1) and (3.2). The *determinant line bundle* associated with the Spin^c -structure via the homomorphism (3.2) is the complex line bundle

$$\mathbf{L} = P \times_{\det} \mathbb{C} \quad (3.4)$$

over M . As P can be thought of as a circle bundle over $SO(E)$, the associated bundle $P \times_{U(1)} \mathbb{C} \rightarrow SO(E)$ is a square root of the pullback to $SO(E)$ of the determinant bundle L .

Existence and obstruction. Existence of Spin^c structures on a principal bundle Q over a space M can be characterized in several equivalent ways (see [73])

1. Q has a Spin^c structure;
2. there exists an S^1 bundle P_1 such that the Stiefel-Whitney class of the fiber product $w_2(Q \times_M P_1) = 0$;
3. there exists an S^1 bundle P_1 such that $w_2(Q) \equiv c_1(P_1) \pmod{2}$;
4. there exists a cohomology class $z \in H^2(M; \mathbb{Z})$ such that $w_2(Q) \equiv z \pmod{2}$.

Then Q has a Spin^c structure if and only if the Stiefel-Whitney class $w_2(Q) \in H^2(M; \mathbb{Z}_2)$ is the \mathbb{Z}_2 -reduction of an integral class $z \in H^2(M; \mathbb{Z})$. Therefore, if $H^2(M; \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z}_2)$ is surjective then every $SO(n)$ -bundle over M admits a Spin^c structure.

3.2 Why Spin^c -structures in eleven dimensions?

The gravitino. Spin^c manifolds are relevant in M-theory and type II string theory. The gravitino (in eleven dimensions) a priori requires a Spin structure since it is a section of a twisted Spin bundle. However, due to the coupling to the C -field, via its field strength G_4 , the fermion can be charged under a $U(1)$ group, whose field strength is the abelian 2-form obtained as a result of the dimensional reduction of G_4 on the torus. Thus, one can define the gravitino on a Spin^c -manifold M provided that one imposes the following consistency condition, $w_2(TM) = 2Qc_1(F)$ and, since the second Stiefel-Whitney class $w_2(TM)$ is the mod 2 reduction of an integral class, c_1 (the first Chern class of the corresponding circle bundle F), the fermion charge Q must be half-integral [62] [36]. The main example is type IIA string theory on $AdS_5 \times \mathbb{CP}^2 \times S^1$ lifting to M-theory on $AdS_5 \times \mathbb{CP}^2 \times T^2$, both spaces being Spin^c but not Spin. The fact that these spaces have Spin^c structure plays an important role in the discussion of the supersymmetry of the Kaluza-Klein massive modes in [111].

Parallel and Killing spinors. Manifolds with Spin^c structure are also relevant in supersymmetric compactification since they admit parallel and Killing spinors, and hence lead to unbroken supersymmetry. For the first, there is the following result [126]. A simply connected Spin^c manifold M carries a parallel spinor if and only if it is isometric to the Riemannian product $M_1 \times M_2$ of a simply connected Kähler manifold M_1 and a simply connected Spin manifold M_2 carrying a parallel spinor, and the Spin^c structure of M is the product of the canonical Spin^c structure of M_1 and the Spin structure of M_2 . For real Killing spinors one has [126]

- The cone over Spin^c manifolds with real Killing spinors inherits a canonical Spin^c structure such that the Killing spinor on the base induces a parallel spinor on the cone.
- Every Sasakian manifold M^{2k+1} carries a canonical Spin^c structure. If M is Einstein then the auxiliary bundle of the canonical Spin^c structure is flat, so if in addition M is simply connected then it is Spin.
- The only simply connected Spin^c manifolds admitting real Killing spinors other than the Spin manifolds are the non-Einstein ² Sasakian manifolds M^{2k+1} .

For example, the 3-sphere S^3 admits two left-invariant non-Einstein Sasakian metrics with scalar curvature $\mathcal{R} = 1 \pm \sqrt{5}$ admitting weak Killing spinors [100].

²Note that a manifold does not necessarily have to be Einstein in order for it to satisfy the low-energy equations of string theory.

Relating Spin^c on Z^{12} and Spin^c on Y^{11} . Spin structures on the bounding manifold Z^{12} have been considered at the end of section 2.5. Now start with Z^{12} being a Spin^c manifold with boundary Y^{11} . Since $TZ^{12}|_{Y^{11}} = TY^{11} \oplus \mathcal{O}$ and \mathcal{O} obviously has a Spin^c structure, then we can induce a Spin^c structure on Y^{11} . Conversely, given Y^{11} a Spin^c eleven-dimensional manifold we can take $Z^{12} = [0, \infty) \times Y^{11}$ and induce a natural Spin^c structure on Z^{12} .

3.3 Relevant examples of eleven-dimensional Spin^c manifolds

3.3.1 Contact eleven-dimensional manifolds

In eleven (and in fact in all odd) dimensions, contact manifolds have a Spin^c structure essentially for the same reason as for the case of almost complex structures, namely that the first obstruction is the third integral Stiefel-Whitney class W_3 . Take the C -field to be of the form $C_3 = \varpi \wedge d\varpi$, where ϖ is a non-closed one-form on Y^{11} . Then the Chern-Simons integrand $C_3 \wedge G_4 \wedge G_4$ in the M-theory action would be $\varpi \wedge (d\varpi)^{\wedge 5}$, which is not zero (as it can be taken to be the volume form). Such a form ϖ is called a contact form and the condition on the above eleven-form defines a contact structure on Y^{11} , i.e. the hyperplane bundle H of TY^{11} given by $H = \ker \varpi$. See for instance [31] for a more detailed description of contact structures. If we take the M-theory circle direction to be along the first factor ϖ in C_3 , then the vector field v evaluated on ϖ is a constant, which we normalize to the value 1. Such a v for which $\varpi(v) = 1$ is called the Reeb field.³ Now a contact metric structure on Y^{11} produces a reduction of the structure group $SO(11)$ of the tangent bundle TY^{11} to the subgroup $U(5)$. Homotopy classes of such reductions, for the case of the sphere, are classified by (see [115]) $\pi_{11}(SO(11)/U(5)) \cong \mathbb{Z}$.

The pair $(H, d\varpi_H)$ is a symplectic vector bundle, and we have an almost complex structure J on H compatible with $d\varpi_H$, so that we get a Hermitian metric on H given by $g_{\varpi, H}(X, Y) = d\varpi(X, JY)$. The trivial extension of the complex structure J to TY^{11} , via $Jv = 0$, leads to an extension of the metric on H to one on TY^{11} called the *Webster metric* by setting

$$g_{\varpi}(X, Y) = d\varpi(X, JY) + \varpi(X)\varpi(Y). \quad (3.5)$$

Note that $g_{\varpi}(v, X) = \varpi(X)$ and $g_{\varpi}(JX, Y) = d\varpi(X, Y)$, for all $X, Y \in TY^{11}$.

Let $\mathcal{V}_{\mathbb{C}}$ be the subbundle of $T^{\mathbb{C}}Y^{11}$ given by $\mathcal{V}_{\mathbb{C}} = \mathbb{C}v$. Then we have a decomposition

$$T^{\mathbb{C}}Y^{11} = T^{1,0}Y^{11} \oplus T^{0,1}Y^{11} \oplus \mathcal{V}_{\mathbb{C}}, \quad (3.6)$$

where $T^{1,0}Y^{11}$ (respectively $T^{0,1}Y^{11}$) is the subbundle given by the eigenspace of the complex extension J on $H^{\mathbb{C}}$ to the eigenvalue i (respectively $-i$). There is an isomorphism

$$\Lambda^{11}T^{\mathbb{C}}Y^{11} = \bigoplus_{p+q+r=11} \Lambda^p T^{1,0}Y^{11} \otimes \Lambda^q T^{0,1}Y^{11} \otimes \Lambda^r \langle v \rangle. \quad (3.7)$$

Note that $\Lambda^r \langle v \rangle = 0$ for $r > 1$. Let us form the following bundles: $\Lambda_H^{p,q}(Y^{11}) := \Lambda^p T^{1,0}(Y^{11})^* \otimes \Lambda^q T^{0,1}(Y^{11})^*$ and then

$$\Lambda^{p,0}(Y^{11}) := \Lambda_H^{p,0}(Y^{11}) \otimes \Lambda_H^{p-1,0}(Y^{11}) \wedge \varpi. \quad (3.8)$$

The top exterior power bundle $K := \Lambda^{6,0}(Y^{11}) = \Lambda_H^{5,0}(Y^{11})$ is called the canonical bundle. Its dual K^{-1} , called the anticanonical bundle, in fact, from [135], serves as the determinant line bundle for a Spin^c structure on Y^{11} . This Spin^c structure is called the canonical Spin^c structure. Any other Spin^c structure with determinant bundle L will have an associated principal Spin^c bundle that differs from the canonical principal Spin^c bundle by tensoring with some $U(1)$ principal bundle. Now let us take this latter bundle to be the M-theory circle bundle. The spinor bundle is then

$$SY^{11,c} = \Lambda_H^{0,*}(Y^{11}) \otimes \mathcal{L}, \quad (3.9)$$

³It is obvious but implicit that the resulting B -field evaluated on v is zero.

where \mathcal{L} is the complex line bundle corresponding to the M-theory circle bundle. Then we have $\mathcal{L}^2 = K \otimes L$. The Clifford multiplication by iv induces the decomposition

$$SY^{11,c} = S^+Y^{11,c} \oplus S^-Y^{11,c} \quad (3.10)$$

into eigenspaces $S^\pm Y^{11,c}$ of iv with eigenvalue ± 1 . There is also the Clifford multiplication by $id\varpi$ which induces the decomposition $SY^{11,c} = \bigoplus_{q=0}^5 S_{5-2q}Y^{11,c}$, where $S_k Y^{11,c}$ is the eigenspace of $id\varpi$ with eigenvalue k . From [135], we have isomorphisms

$$S^\pm Y^{11,c} \cong \Lambda_H^{0,\text{even/odd}}(Y^{11}) \otimes \mathcal{L} \quad (3.11)$$

$$S_{5-2q}Y^{11,c} \cong \Lambda_H^{0,q}(Y^{11}) \otimes \mathcal{L}. \quad (3.12)$$

The B -field and the Kähler form. In the above example, the resulting B -field will be $d\varpi$. So the Clifford multiplication leading to (3.12) is really multiplication by (the lift of) iB from the symplectic manifold whose tangent bundle is H . The correspondence is provided by the Boothby-Wang theorem (cf. [31]): every compact, regular contact manifold is a principal circle bundle over a symplectic manifold whose symplectic form determines an integral cocycle. The appearance of i multiplying B is compatible with the way it appears as part of the complexified Kähler form $\omega + iB$ in the Kähler case.

The Chern-Simons form. The condition $\varpi \wedge (d\varpi)^5 \neq 0$, from the beginning of this section, is independent of the choice of ϖ and is a property of $v = \ker \varpi$. Any other 1-form defining the same hyperplane field must be of the form $\lambda\varpi$ for some smooth function $\lambda : Y^{11} \rightarrow \mathbb{R}^\times$ so that

$$(\lambda\varpi) \wedge (d(\lambda\varpi))^5 = \lambda\varpi \wedge (\lambda d\varpi + d\lambda \wedge \varpi)^5 = \lambda^6 \varpi \wedge (d\varpi)^5 \neq 0, \quad (3.13)$$

as this is proportional to the volume form. In this case we see that the sign of this volume form depends only on v and not on the choice of ϖ , so the contact structure v induces a natural orientation of Y^{11} . We assume Y^{11} to be oriented with a specific orientation so that we can have positive and negative contact structures depending on the relative sign. With $C_3 = \pm\varpi \wedge d\varpi$, and using $dC_3 = G_4$, we get for the Chern-Simons term $\frac{1}{6}C_3 \wedge G_4 \wedge G_4 = \pm\frac{1}{6}\varpi \wedge (d\varpi)^5$, which is proportional to the volume form with a possible sign difference. If we take the standard volume form to be $\text{vol}_{\text{st}} = \frac{1}{5!}\varpi \wedge (d\varpi)^5$ then the Chern-Simons term is $\pm 20\text{vol}_{\text{st}}$, a cosmological ‘constant’.

3.3.2 Spherical space forms

The representation ϱ from section 2.2.1 can always be lifted to $\text{Spin}^c(2k)$, so that $M(\Gamma, \varrho)$ is Spin^c . Assume that there exists a fixed-point-free representation ϱ of a finite group Γ in the unitary group $U(k)$, for $k \geq 2$. Let $M = M(\varrho) := S^{2k-1}/\varrho(\Gamma)$ be the resulting quotient manifold. The stable tangent bundle $TM \oplus 1$ is naturally isomorphic to the underlying real vector bundle of the complex vector bundle defined by ϱ over M and thus TM admits a stable almost complex structure. There exists a unique group homomorphism f from $U(k)$ to $\text{Spin}^c(2k)$, so that the associated determinant line bundle is $\det(\varrho)$. This provides M with a natural Spin^c structure. This structure can be reduced to a Spin structure iff there exists a square root of the linear representation $\det(\varrho)$. We will use the following in section 6.2.

1. If $|\Gamma|$ is odd then the square root of any linear representation exists.
2. If $|\Gamma|$ is even then the square root of $\det(\varrho)$ exists iff k is even. This implies that the dimension of M is congruent to 3 mod 4. In this case, there are inequivalent Spin structures on M and the choice of a Spin structure is equivalent to a choice of a square root of $\det(\varrho)$.

Projective Spaces. In addition to Spin manifolds being Spin^c , there are of course Spin^c manifolds that do not have a Spin structure. Let L be the nontrivial flat line bundle over \mathbb{RP}^m given by $L := S^m \times \mathbb{R} / \sim$, with the identification $(\xi, x) \sim (-\xi, -x)$. Let $x := w_1(L) \in H^1(\mathbb{RP}^m; \mathbb{Z}_2)$. The integral second cohomology group is $H^2(\mathbb{RP}^m; \mathbb{Z}) = x^2 \cdot \mathbb{Z}_2$. The bundle $rL := L \oplus \cdots \oplus L$, r times, has Stiefel-Whitney class $w(rL) = (1+x)^r$, so that $w_1(rL) = rx$ and $w_2(rL) = \frac{1}{2}r(r-1)x^2$. We see that rL admits a Spin^c structure for $r = 4k+2$, since for these values $w_2 = x^2$ lifts to $H^2(\mathbb{RP}^2; \mathbb{Z})$. Since $T(\mathbb{RP}^m) \oplus 1 = (m+1)L$, we see that \mathbb{RP}^m admits a Spin^c structure for $m = 4k+1$. Therefore, in our range of dimensions we have: $\mathbb{RP}^1 = S^1$, \mathbb{RP}^5 and \mathbb{RP}^9 . We get eleven-dimensional Spin^c manifolds by taking products (or bundles) with the above projective spaces. The Stiefel-Whitney classes satisfy

$$w_2(X \times Y) := w_2(TX \oplus TY) = w_2(X) + w_2(Y) + w_1(X)w_1(Y), \quad (3.14)$$

so that for oriented manifolds, and with one of the factors being Spin and another being Spin^c , the product will be Spin^c . For the case of bundles, let $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 1$ be a short exact sequence of real vector bundles. Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. If V_i and V_j admit Spin^c structures then V_k admits a natural Spin^c structure and the canonical line bundles corresponding to the vector bundles satisfy $L(V_i) = L(V_j) \otimes L(V_k)$.

Example: Lens spaces (and their bounding spaces). Consider the lens spaces from section 2.2.1. Choose an integer p and an n -tuple of integers $\vec{a} = (a_1, \dots, a_n)$ coprime to p . The quotient of the $(2n-1)$ -dimensional sphere $S^{2n-1} \subset \mathbb{C}^{2n}$ by the \mathbb{Z}_p action on \mathbb{C}^{2n} is given by

$$\zeta \cdot (z_j) = (\zeta^{a_j} z_j) \quad \text{with} \quad \zeta^p = 1. \quad (3.15)$$

The resulting quotient space is the $(2n-1)$ -dimensional lens space $L^{2n-1}(p, \vec{a})$. The lens space $L^{2n-1}(p, \vec{a})$ is diffeomorphic to the boundary of the $2n$ -manifold $Z^{2n}(p, \vec{a})$, which is the quotient of the $2n$ -dimensional ball $\mathbb{B}^{2n} \subset \mathbb{C}^{2n}$ by the \mathbb{Z}_p action. The space $Z^{2n}(p, \vec{a})$ has a resulting isolated singularity at the origin $[(0, \dots, 0)]$.

3.4 Multiple Spin^c structures

The set of Spin^c structures. The group $H^2(M; \mathbb{Z}) = \text{Vect}_1^{\mathbb{C}}(M) = \text{Prin}_{U(1)}(M)$ of isomorphism classes of principal $U(1)$ bundles over M acts transitively and effectively on the set $\text{Spin}^c(\zeta)$ of isomorphism classes of Spin^c structures on the bundle ζ . This is analogous to the Spin case, considered in section 2.1.

Given two Spin^c structures σ_1^c and σ_2^c we can define their difference as the unique line bundle L via $\sigma_2^c = \sigma_1^c \otimes L$, so that the collection of Spin^c structures is (non-canonically) isomorphic to $H^2(M; \mathbb{Z})$. This is an $H^2(M; \mathbb{Z})$ -torsor, i.e. an affine space modeled on $H^2(M; \mathbb{Z})$ in the sense that the difference between two Spin^c structures is an element in $H^2(M; \mathbb{Z})$ but there is no distinguished origin of this space.

Example: Lens space and its bounding space. ⁴ Any Spin^c structure σ^c on $Z^{12}(p, \vec{a})$ can be obtained by twisting the canonical Spin^c structure σ_{can}^c by the unique line bundle L on $Z^{12}(p, \vec{a})$. Any topological line bundle L on $Z^{12}(p, \vec{a})$ can be obtained as the quotient of the trivial bundle $\mathbb{B}^{12} \times \mathbb{C}$ by a \mathbb{Z}_p action given by

$$\zeta_p^l \cdot (z, u) = (\zeta_p^l z, \varrho(\zeta_p)^l \cdot u) \quad (3.16)$$

for some $U(1)$ representation $\varrho : \mathbb{Z}_p \rightarrow U(1)$ of \mathbb{Z}_p . Now a Spin^c structure on a manifold M induces one on $M \times \mathbb{R}$ and vice versa. Then there is a one-to-one correspondence between the set of all Spin^c structures $\text{Spin}^c(Z^{12}(p, \vec{a}))$ on $Z^{12}(p, \vec{a})$ and the set of representations $R(\mathbb{Z}_p, U(1)) \cong \mathbb{Z}_p$ from \mathbb{Z}_p to $U(1)$, which we can see as follows [75]. The set $\text{Spin}^c(L(p; \vec{a}))$ of all Spin^c structures on $L(p; \vec{a})$ is in one-to-one correspondence with the set $\text{Pic}^t(L(p; \vec{a}))$ of all topological line bundles on $L(p; \vec{a})$. Similarly, the set $\text{Spin}^c(Z^{12}(p; \vec{a}))$ of all

⁴The following discussion can be given in more general terms; however, we are interested in the application to our specific dimensions.

Spin^c structures on $Z^{12}(p; \vec{a})$ is in one-to-one correspondence with the set $\text{Pic}^t(Z^{12}(p; \vec{a}))$ of all topological line bundles on $Z^{12}(p; \vec{a})$. The idea is to show that the map $i^* : \text{Pic}^t(Z^{12}(p; \vec{a})) \rightarrow \text{Pic}^t(L(p; \vec{a}))$, induced from the inclusion $i : L(p; \vec{a}) \hookrightarrow Z^{12}(p; \vec{a})$, is bijective. Any line bundle in $\text{Pic}^t(Z^{12}(p; \vec{a}))$ is by definition the quotient of $\mathbb{B}^{12} \times \mathbb{C}$ divided by some \mathbb{Z}_p -action with representation $\mathbb{Z}_p \rightarrow U(1)$. Hence $\text{Pic}^t(Z^{12}(p; \vec{a})) \cong \mathbb{Z}_p$. On the other hand, $\text{Pic}^t(L(p; \vec{a})) \cong H^2(L(p; \vec{a})) \cong \mathbb{Z}_p$. Now consider any $L \in \text{Pic}^t(L(p; \vec{a}))$ with connection A and curvature F_A . The first Chern class $c_1(L)$ is torsion, i.e. $c_1(L) \in H^2(L(p; \vec{a})) \cong \mathbb{Z}_p$. By Chern-Weil theory, the de Rham cohomology class $c_1(L) = [-F_A]$ vanishes, so that $F_A = da$ for some 1-form a on $L(p; \vec{a})$. This gives that $A - a$ is a flat connection. The holonomy $\varrho : \mathbb{Z}_p \rightarrow U(1)$ of this connection gives an isomorphism of L with the quotient of $S^{12} \times \mathbb{C}$ by the \mathbb{Z}_p action given by the holonomy ϱ . Then the line bundle $(\mathbb{B}^{12} \times \mathbb{C})/\mathbb{Z}_p$ corresponds to L , thus establishing surjectivity of i^* . Since $\text{Pic}^t(L(p; \vec{a}))$ and $\text{Pic}^t(Z^{12}(p; \vec{a}))$ are isomorphic to \mathbb{Z}_p and have the same order, the map i^* is bijective. Therefore, the set of all Spin^c structures on $L(p; \vec{a})$ can be parametrized by

$$\mathbb{Z}_p \cong R(\mathbb{Z}_p, U(1)) \cong \text{Spin}^c(Z^{12}(p; \vec{a})) \cong \text{Spin}^c(L(p; \vec{a})) . \quad (3.17)$$

3.5 Spin and Spin^c cobordism and extension to twelve dimensions

Spin^c characteristic classes. The map $B\text{Spin}^c \rightarrow BSO \times BU(1)$ is an isomorphism on rational cohomology. One has Pontrjagin classes $p_i \in H^{4i}(B\text{Spin}^c; \mathbb{Q})$ and the class $c_1 \in H^2(B\text{Spin}^c; \mathbb{Q})$ of the canonical bundles so that one has [156] the ring $H^*(B\text{Spin}^c; \mathbb{Q}) = \mathbb{Q}[c_1, p_i, e]$. Every $\text{Spin}^c(n)$ -bundle over a manifold is a pullback of the universal $\text{Spin}^c(n)$ -bundle $E\text{Spin}^c(n) \rightarrow B\text{Spin}^c(n)$ through the map to the classifying space. The class e corresponds to the Euler class of the bundle E , and c_1 corresponds to the first Chern class of the determinant line bundle L in (3.4).

Cobordism. Let F denote either Spin or Spin^c . Consider two 11-dimensional manifolds Y_1^{11} and Y_2^{11} . A sum operation $Y_1^{11} + Y_2^{11}$ is defined by disjoint union and $-Y^{11}$ is defined by reversing the orientation of Y^{11} and by taking the appropriate F structure. We say that Y_1^{11} is F -bordant to Y_2^{11} if there exists a smooth compact manifold Z^{12} so that $\partial Z^{12} = Y_1^{11} - Y_2^{11}$, and so the F -structure on ∂Z^{12} extends over Z^{12} . Bordism defines an equivalence relation.

1. Since $\partial(Y^{11} \times I) = Y^{11} + (-Y^{11})$, Y^{11} is bordant to itself.
2. Reversing orientation of Z^{12} gives that Y_1^{11} is bordant to Y_2^{11} , implying that Y_2^{11} is bordant to Y_1^{11} .
3. Let $\partial(Z_1^{12}) = Y_1^{11} - Y_2^{11}$ and $\partial(Z_2^{12}) = Y_2^{11} - Y_3^{11}$, so that Y_1^{11} is bordant to Y_2^{11} and Y_2^{11} is bordant to Y_3^{11} . We can find a collar neighborhood of Y_2^{11} in Z_1^{12} and form $Y_2^{11} \times (-\epsilon, 0]$ and a collar neighborhood of Y_2^{11} in Z_2^{12} of the form $Y_2^{11} \times [0, \epsilon)$ for some $\epsilon > 0$. Gluing Z_1 to Z_2^{12} along $Y_2^{11} \times \{0\}$ creates a smooth manifold Z_3^{12} with a natural F structure which provides the desired bordism from Y_1^{11} to Y_3^{11} .

Let Ω_{11}^F be the set of equivalence classes. Disjoint union makes Ω_{11}^F into an abelian group and $\partial(Y^{11} \times I) = Y^{11} + (-Y^{11})$, where $-Y^{11}$ is the inverse of Y^{11} . Let Ω_*^F be the associated graded abelian group. This is a ring under Cartesian product. The forgetful functor defines ring morphisms

$$\Omega_*^{\text{Spin}} \longrightarrow \Omega_*^{\text{Spin}^c} , \quad \Omega_*^{\text{Spin}} \longrightarrow \Omega_*^{SO} , \quad \Omega_*^{\text{Spin}^c} \longrightarrow \Omega_*^{SO} . \quad (3.18)$$

The characteristic classes define characteristic numbers which are bordism invariants.

1. *The O characteristic numbers:* The Stiefel-Whitney classes $w_i \in H^i(-; \mathbb{Z}_2)$.
2. *The SO characteristic numbers:* The Pontrjagin classes $p_i \in H^{4i}(-; \mathbb{Q})$.
3. *The Spin characteristic numbers:* The KO -characteristic numbers and the Stiefel-Whitney numbers.

4. *The Spin^c characteristic numbers:* These are the Chern-Pontrjagin numbers and the Stiefel-Whitney numbers. Define the Chern-Pontrjagin classes as follows. Let $\det : \text{Spin}^c \rightarrow U(1)$ define a line bundle $L \in \text{Vect}_{\mathbb{C}}(B\text{Spin}^c)$. Then $H^*(B\text{Spin}^c; \mathbb{Z})/\text{torsion} = \mathbb{Z}[c_1(L), p_i]$. There is a natural evaluation (see [78])

$$\mu : \Omega^{\text{Spin}^c}/\text{torsion} \otimes \mathbb{Z}[c_1(L), p_i] \rightarrow \mathbb{Z}. \quad (3.19)$$

Some properties of Spin cobordism.

1. Every Spin cobordism class is represented by a connected manifold.
2. For $n \geq 3$ every Spin cobordism class is represented by a simply-connected manifold.
3. For $n \geq 5$ every Spin cobordism class is represented by a 2-connected manifold.
4. For $n \geq 1$, $\Omega_n^{\text{Spin}} \cong \Omega_{n+8}^{\text{Spin}}$, so there is a mod 8 periodicity in the Spin cobordism groups.
5. $\Omega_n^{\text{Spin}} = 0$ for $n = 3, 5, 6, 7 \bmod 8$.
6. For $n = 0$: $\Omega_0^{\text{Spin}} \cong \mathbb{Z}$.
7. For $n = 1$: A Spin structure is defined to be a 2-fold covering of $PSO(S^1) = S^1$. Then $\Omega_1^{\text{Spin}} \cong \mathbb{Z}_2$, generated by S^1 with $2S^1 = 0$. See section 2.1.
8. For $n = 2$: $\Omega_2^{\text{Spin}} \cong \mathbb{Z}_2$, generated by $S^1 \times S^1$ with $2(S^1 \times S^1) = 0$.
9. For $n = 4$: $\Omega_4^{\text{Spin}} \cong \mathbb{Z}$, generated by the Kummer surface $V^2(4)$, which is the variety of degree 4 in $\mathbb{C}P^3$ with vanishing first Chern class.
10. For $n = 8$: $\Omega_8^{\text{Spin}} \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by $\mathbb{H}P^2$ and a manifold L^8 such that $4L^8$ is Spin cobordant to $V^2(4) \times V^2(4)$, the product of two Kummer surfaces.
11. To study Spin cobordism invariant quantities it is enough to check them against the corresponding generators, since any other Spin manifold of a given dimension is ‘built’ out of such generators.

Spin^c-structures on Y^{11} . Here we apply the construction in [22] [23] and continue the discussion from section 2.6. In the non-equivariant case, σ_Y induces a Spin structure on $\pi^* P_{SO}(X^{10})$ which gives an equivariant Spin^c structure σ_Y^c if we endow $\xi(\sigma_Y^c) : Y^{11} \times S^1 \rightarrow Y^{11}$ with the diagonal action of S^1 on $Y^{11} \times S^1$. The canonical $U(1)$ bundle of the quotient Spin^c structure σ_X^c on X^{10} of σ_Y^c if $\xi(\sigma_X^c) = \xi(\sigma_Y^c)/S^1 = -\pi$, i.e. π with the $U(1)$ -action reversed. Therefore σ_X^c induces a Spin^c structure σ_Z^c on Z^{12} with $\xi(\sigma_Z^c) = \pi \otimes (-\pi)$ trivial. Given a connection ω^π on π we also get an induced Spin^c -Dirac structure (σ_X^c, ω^X) on X^{10} with $\xi(\sigma_X^c) = -\pi$ and connection $\omega_X = -\omega^\pi$. In this case Z^{12} is a Spin manifold and we must have $w_2(X^{10}) = c_1(\pi) \bmod 2$. Thus the set $\text{Spin}(Y^{11})$ of isomorphism classes of Spin structures on Y^{11} is given by

1. X^{10} is Spin and π is not: $\text{Spin}(Y^{11}) = \pi^* \text{Spin}(X^{10})$.
2. Both X^{10} and π are not Spin: $\text{Spin}(Y^{11}) = \partial \text{Spin}(Z^{12})$.

We consider two Spin^c structures on a fixed equivariant Spin^c structure σ_Y^c on Y^{11} . Such Spin^c structures on Y^{11} are obtained from Spin^c structures on X^{10} and vice-versa. A Spin^c structure $\sigma_X^c : P_{\text{Spin}^c}(X^{10}) \rightarrow P_{SO}(X^{10})$ on X^{10} induces a Spin^c structure $\sigma_Y^c : P_{\text{Spin}^c}(Y^{11}) \rightarrow P_{SO}(Y^{11})$ on Y^{11} . The equivariant Spin^c structure σ_Y^c extends to a Spin^c structure σ_Z^c on the disk bundle Z^{12} which is induced from the Spin^c structure σ_X^c and the canonical Spin^c structure $\sigma_\pi^c : P_{\text{Spin}^c}(\pi) = Y^{11} \times_{U(1)} \text{Spin}^c(2) \rightarrow Y^{11} = P_{SO}(\pi)$ of the principal S^1 bundle π , $\sigma_Z : P_{\text{Spin}^c}(Z^{12}) \rightarrow P_{SO}(Z^{12})$, where

$$P_{\text{Spin}^c}(Z^{12}) = \pi^* (P_{\text{Spin}^c}(X^{10}) \times_{X^{10}} P_{\text{Spin}^c}(\pi)) \times_{\text{Spin}^c(10) \times \text{Spin}^c(2)} \text{Spin}^c(12) \quad (3.20)$$

$$P_{SO}(Z^{12}) = \pi^* (P_{SO}(X^{10}) \times_{X^{10}} P_{SO}(\pi)) \times_{SO(10) \times SO(2)} SO(12). \quad (3.21)$$

Its canonical bundle is $\xi(\sigma_Z^c) = \pi_D^*(\xi(\sigma_X^c) \otimes \pi)$. Putting $\omega_Z = \pi_D^*(\omega_X^c \otimes \omega^\pi)$ we get an equivariant Spin^c -Dirac structure (σ_Z^c, ω_Z) on Z^{12} . Restriction to Y^{11} of such Spin^c -Dirac structures is *boundary Spin^c structures*. They are equivariant but not strictly equivariant.

Extension to twelve dimensions for Spin^c We need to check that the Spin^c eleven-dimensional manifold extends to twelve dimensions. We also need to check that the corresponding E_8 bundle also extends. The former involves checking whether and when the cobordism group $\Omega_{11}^{\text{Spin}^c}$ is zero, and the latter involves $\Omega_{11}^{\text{Spin}^c}(K(\mathbb{Z}, 4))$, since in our range of dimensions, E_8 has the homotopy type of the Eilenberg-MacLane space $K(\mathbb{Z}, 3)$, whose classifying space is then of type $K(\mathbb{Z}, 4)$.

Rationally [156],

$$\Omega_{11}^{\text{Spin}^c} \otimes \mathbb{Q} \cong \Omega_{11}(K(\mathbb{Z}, 2)) \otimes \mathbb{Q} \cong \Omega_{11}^{\text{Spin}}(K(\mathbb{Z}, 2)) \otimes \mathbb{Q} \quad (3.22)$$

3.6 The APS index in the Spin^c case

The complex representations of $\text{Spin}^c(n)$ are the same as those of $\text{Spin}(n)$. So, when n is even we have a \mathbb{Z}_2 -graded Spin bundle $S = S^+ \oplus S^-$. The grading is provided by the chirality operator Υ , which is defined by the Clifford multiplication $i^{n/2}e_1 \cdots e_n$, where e_i constitute a local orthonormal frame for TZ^{12} .

The Spin^c Dirac operator is defined in the same way as the Spin Dirac operator, i.e. by composing the covariant derivative with Clifford multiplication

$$D^c = \sum_{i=1}^n e_i \cdot \nabla_{e_i} : \Gamma(Z^{12}, S) \rightarrow \Gamma(Z^{12}, S), \quad (3.23)$$

which decomposes as $\begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}$ since D^c anticommutes with Υ . Here D^\pm are the restrictions of D^c to S^\pm . Unitarity of ∇ implies that D^\pm is the adjoint of D^\mp . The index of the Spin^c Dirac operator is $\text{Ind}(D^c) = \dim \ker D^+ - \dim \ker D^-$. Given a Hermitian vector bundle E over Z^{12} with unitary connection ∇^E , the twisted Spin^c Dirac operator is $D_E^c : \Gamma(Z^{12}, S \otimes E) \rightarrow \Gamma(Z^{12}, S \otimes E)$. Assuming Z^{12} has boundary $\partial Z^{12} = Y^{11}$, the restriction of the Spin^c Dirac operator to the boundary is

$$D_{Y,E}^c : \Gamma((S \otimes E)|_{Y^{11}}) \rightarrow \Gamma((S \otimes E)|_{Y^{11}}), \quad (3.24)$$

which is formally self-adjoint and elliptic. Imposing global APS boundary conditions and requiring product type structures near the boundary, the Atiyah-Patodi-Singer index theorem states that (see [107])

$$\text{Ind}_{\text{APS}}(D_E^c) = \int_{Z^{12}} \widehat{A}(\nabla^{TZ, LC}) \text{ch}(\nabla^{L^{1/2}}) \text{ch}(\nabla^E) - \xi(D_{Y,E}^c), \quad (3.25)$$

where

$$\xi(D_{Y,E}^c) = \frac{1}{2} \eta(D_{Y,E}^c) + \frac{1}{2} \dim \ker(D_{Y,E}^c). \quad (3.26)$$

3.7 Effect of the Spin^c condition on the C -field

Here we consider the M2-brane and the M5-brane partition functions in the Spin^c case. Consider first the Stiefel-Whitney classes in the presence of the Spin^c line bundle with first Chern class c .

1. $w_2(M) \equiv c \pmod{2}$, so that the manifold M is Spin if c is even. Hence in order for M to be only Spin^c , the first Chern class c has to be odd.
2. The fourth Stiefel-Whitney class satisfies

$$w_4(M) \equiv \frac{c^2 - p_1(M)}{2} \pmod{2}. \quad (3.27)$$

We see that if c is even, i.e. when M is Spin, then we are back to the condition $w_4(M) \equiv \frac{1}{2}p_1 \pmod{2}$. However, when M is only Spin^c , so that c is odd, $p_1(M)$ can also be odd and still we could have $w_4 = 0$.

The M5-brane partition function. The expression (3.27) is used in the M5-brane partition function (section 5.2 of [166]). Consider the M5-brane on a six-manifold W^6 . Take a product with a circle and extend $W^6 \times S^1$ to a bounding Spin^c 8-manifold M^8 , $\partial M^8 = W^6 \times S^1$. Witten makes use of the Spin^c structure on W^6 to formulate the partition function. The phase is given by the expression

$$\Omega_{W^6}(x) := \exp \left[i\pi \int_{M^8} (z^2 + \lambda z) \right], \quad (3.28)$$

where $2\lambda = p_1 - c^2$ and the reduction modulo 2 of c is $r_2(c) = w_2(M^8)$.

The M2-brane partition function. The Spin^c index on a manifold M coincides with the \widehat{A} -genus if $c_1(L)$ is torsion and coincides with the index of the Dirac operator if the Spin^c structure is induced from a Spin structure on M . Thus, in order to go beyond the standard Spin discussions we have to take $c = c_1(L)$ not to be torsion and in particular to be odd. When Y^{11} is not Spin, $\lambda = \frac{1}{2}p_1$ is not guaranteed to be divisible by two, so that the quantization condition $G_4 + \frac{1}{4}p_1(Y^{11}) \in H^4(Y^{11}; \mathbb{Z})$ of [165] would require careful consideration. In particular, G_4 cannot be set to zero. We now consider the Spin^c case, i.e. take the membrane worldvolume to wrap/be a three-manifold with a Spin^c structure. Let D^c be the corresponding Dirac operator and \mathbb{L} the corresponding determinant line bundle. The effective action involves two factors:

1. The ‘topological term’ $\exp i \int_{M^3} C_3$,
2. The fermion term $\exp i \int_{M^3} \bar{\psi} D^c \psi$.

The first factor will simply give $\exp i \int_{X^4} G_4$. Now consider the second factor. Corresponding to the map $\phi : X^4 \rightarrow Z^{12}$ we have the index of the Spin^c Dirac operator for spinors that are sections of $S^c(X^4) \otimes \phi^* T Z^{12}$ given via the index theorem by the degree two expression

$$\begin{aligned} \text{Ind} D^c &= \left[\int_{X^4} \widehat{A}(X^4) \wedge \text{Ph}(\phi^* T Z^{12}) e^{\frac{1}{2}c_1(\mathbb{L})} \right]_{(4)} \\ &= \int_{X^4} \left[1 - \frac{1}{24} p_1(T X^4) \right] [\text{rank}(T Z^{12})|_{X^4} + \text{Ph}_2(\phi^* T Z^{12})] \left[1 + \frac{1}{8} c_1^2(\mathbb{L}) \right] \\ &= \int_{X^4} \frac{1}{2} p_1(\phi^* T Z^{12}) - \frac{1}{2} p_1(X^4) + \frac{1}{2} c_1^2(\mathbb{L}). \end{aligned} \quad (3.29)$$

Here Ph is the Pontrjagin character, which is the composition of the Chern character of a complex bundle with the realification map. On the other hand we have a decomposition of the restriction of the tangent bundle $T Z^{12}$ to X^4 as $T Z^{12} = T X^4 \oplus N X^4$, with $N X^4$ the normal bundle of X^4 in Z^{12} . Taking the characteristic class λ of both sides of the above decomposition we get that the index is equal to $\lambda(N X^4)$. The effective action involves the square root of the index so that the contribution from the second factor in the effective action is

$$\exp 2\pi i \left[\frac{1}{2} \lambda(N X^4) + \frac{1}{4} c_1^2(\mathbb{L}) \right]. \quad (3.30)$$

Now taking $N(M^3 \hookrightarrow Y^{11}) \cong N(X^4 \hookrightarrow Z^{12})$, we get

1. If M^3 admits a Spin structure then, assuming that Y^{11} admits a Spin structure, then the normal bundle does and so λ is divisible by 2. Also, $c_1(\mathbb{L})$ is even in this case, so that $\frac{3}{2}c_1^2(\mathbb{L})$ is integral. Hence the whole expression is integral and we are essentially back to the result of [165].
2. Now if both Y^{11} and M^3 are only Spin^c , then we get the quantization condition

$$G_4 + \frac{1}{2} \lambda + \frac{1}{4} c_1^2(\mathbb{L}) \in H^4(Y^{11}; \mathbb{Z}). \quad (3.31)$$

This is the condition in the Spin^c case. In [54], such a condition is essentially and implicitly obtained using transformation properties of the C -field in the presence of the RR field, defined via the line bundle with first Chern class c .

3.8 The phase of M-theory in the Spin^c case.

We start by recalling the setting and the results that we need from [54]. The partition function of the C -field and the Rarita-Schwinger field is factored into a modulus and a phase. These can be studied in regards to any of the three dimensions: eleven, as well as twelve and ten. The two bundles that play an important role are the E_8 vector bundle E (and subbundles thereof) and the Rarita-Schwinger bundle, i.e. the Spin bundle $S(Y^{11})$ coupled to the virtual tangent bundle $TY^{11} - 3\mathcal{O}$. Thus, corresponding to the two bundles there are spinors and hence Dirac operators D acting on them. Let h be the number of zero modes of D .

Now, suppose that Y^{11} is the boundary of a Spin manifold Z^{12} , over which any data, such as an E_8 bundle, used in defining D are extended via cobordism arguments, and let $I(D)$ be the index of the extended operator D on Z^{12} , defined with Atiyah-Patodi-Singer (APS) global boundary conditions. Let I_{E_8} be the index of the Dirac operator D_E on Z^{12} , coupled to the E_8 bundle E , with APS global boundary conditions. Let I_{RS} be the index of the Rarita-Schwinger operator D_{RS} i.e. of the Spin bundle coupled to the virtual tangent bundle $TZ^{12} - 4\mathcal{O}$. Denote the latter by $\Phi(Z^{12})$. Then [54]

1. $\Phi(Z^{12})$ is independent of Z^{12} when Z^{12} has no boundary.
2. $\Phi(Z^{12})$ is independent of Z^{12} when Z^{12} has a boundary.

The relation back to eleven dimensions is provided by the APS index theorem. This asserts that $I(D) = \int_Z i_D - \frac{1}{2}h + \eta$, where i_D is the twelve-form whose integral on a closed twelve-manifold would equal $I(D)$. Then the phase of the effective action on Y^{11} is given by (1.1). Assuming a supersymmetric (nonbounding) Spin structure on S^1 , the index of the Dirac operator coupled to $E \otimes \mathcal{L}^k$ is [54]

$$\text{Ind}(D_{E \otimes \mathcal{L}^k}) = \int_{X^{10}} \widehat{A}(X^{10}) \text{ch}(E) e^{kc}, \quad (3.32)$$

where c is the first Chern class $c_1(\pi)$ of the circle bundle $\pi : Y^{11} \rightarrow X^{10}$, and k is the mode number corresponding to powers of the associated line bundle \mathcal{L}^k as $c_1(\mathcal{L}^k) = kc_1(\mathcal{L})$. In what follows we interpret this as corresponding to Spin^c structures and study the corresponding expressions for the phase.

The data in the Spin^c case. The setting we have is the following:

1. Y^{11} is identified with the boundary of the disk bundle Z^{12} .
2. g_Z is a metric on Z^{12} such that $Y^{11} \times I_\epsilon$ with the product metric $g_Y \oplus dt^2$ isometric to a collar neighborhood of (Z^{12}, g_Z) for some $\epsilon > 0$.
3. Let (σ_Y^c, ω_Y) be a strictly equivariant or a boundary Spin^c structure on Y^{11} and Ω_Y the curvature form of ω_Y .
4. $(X^{10}, \sigma_X^c, \omega_X)$ is the induced Spin^c -Dirac structure on X^{10} . Ω is the curvature of ω_X if ω_Y is strictly equivariant and the curvature of $\omega_X \otimes \omega^\pi$ otherwise.
5. If (σ_Y^c, ω_Y) is an equivariant boundary Spin^c -Dirac structure. We have $\omega_Y = \pi^*\Omega$.
6. Now we introduce structure on the E_8 vector bundle E . This is a vector bundle over X^{10} equipped with a connection ∇^E with curvature form Ω^E .
7. ∇^Y is the induced connection over Y^{11} and ∇^Z is a connection on the bundle π^*E over Z^{12} , and (σ_Z^c, ω_Z) is a Spin^c -Dirac structure on Z^{12} extending (σ_Y^c, ω_Y) to Z^{12} .
8. D_{π^*E} is the Spin^c -Dirac operator on $(Z^{12}, \sigma_Z^c, \omega_Z)$ twisted by the coefficient bundle (π^*E, ∇^Z) acting on spinors over Z^{12} satisfying the APS boundary conditions. The twisted Dirac operator $D_{\pi^*E}^Y$ is the corresponding operator on Y^{11} .

3.8.1 The adiabatic limit when (σ_Y^c, ω_Y) is a boundary Spin^c -Dirac structure

In this case, applying the results of [22] [23] [168] [169] gives

$$\lim_{t \rightarrow 0} \overline{\eta}(D_{\pi^*E^Y, g_Y^t}) = \left\langle \widehat{A}(X^{10}) e^{c_1(\sigma_X^c)/2} \text{ch}(E) \left(\frac{e^{c_1(\pi)/2}}{2 \sinh(c_1(\pi)/2)} - \frac{e^{c_1(\pi)/2}}{c_1(\pi)} \right), [X^{10}] \right\rangle \mod \mathbb{Z}. \quad (3.33)$$

Let a be the characteristic class of the E_8 bundle E . Then the Chern character of E is expanded, up to dimension eight, as

$$\text{ch}(E) = 248 + 60a + 6a^2. \quad (3.34)$$

What we are really doing is looking at the Chern character of the real bundle, i.e. the Pontrjagin character $\text{Ph}(E)$. The \hat{A} -genus is expanded in degree $4k$ terms as $\hat{A} = 1 + \hat{A}_4 + \hat{A}_8$, where

$$\hat{A}_4 = -\frac{1}{2^3 \cdot 3} p_1, \quad \hat{A}_8 = -\frac{7p_1^2 - 4p_2}{2^7 \cdot 3^2 \cdot 5}. \quad (3.35)$$

The expansion of the expression of the adiabatic limit (3.33) gives

$$\begin{aligned} \lim_{t \rightarrow 0} \overline{\eta}(D_{\pi^* E^Y, g_Y^t}) &= \frac{248}{2^8 \cdot 3} \left[\frac{11c_1(\pi)^5}{2^2 \cdot 3 \cdot 5 \cdot 7} - \frac{c_1(\pi)^4 c_1(\sigma_X^c)}{2 \cdot 5} - \frac{23c_1(\pi)^3 c_1(\sigma_X^c)^2}{2^2 \cdot 3 \cdot 5} - \frac{c_1(\pi)^2 c_1(\sigma_X^c)^3}{3} - \frac{c_1(\pi) c_1(\sigma_X^c)^4}{2^2 \cdot 3} \right] \\ &- \frac{1}{2^3} \left(\frac{248}{2^2 \cdot 3} \hat{A}_4(X^{10}) + 5a \right) \left[\frac{c_1(\pi) c_1(\sigma_X^c)^2}{2} + c_1(\pi)^2 c_1(\sigma_X^c) + \frac{23c_1(\pi)^3}{2^2 \cdot 3 \cdot 5} \right] \\ &- \frac{1}{2^2} \left[\left(a^2 + \frac{248}{2 \cdot 3} \hat{A}_8(X^{10}) \right) c_1(\pi) \right] \end{aligned} \quad (3.36)$$

Special cases. 1. When the circle bundle π is trivial, $c_1(\pi) = 0$, then the adiabatic limit is zero.

2. When the Spin^c structure is trivial, i.e. $c_1(\sigma_X^c) = 0$, the right-hand side of the expression simplifies to

$$\frac{31 \cdot 11}{2^7 \cdot 3^2 \cdot 5 \cdot 7} c_1(\pi)^5 - \frac{23}{2^5 \cdot 3 \cdot 5} \left(\frac{248}{2^2 \cdot 3} \hat{A}_4(X^{10}) + 5a \right) c_1(\pi)^3 - \frac{1}{2^2} \left[\left(a^2 + \frac{248}{2 \cdot 3} \hat{A}_8(X^{10}) \right) c_1(\pi) \right] \quad (3.37)$$

Example. Consider the ten-dimensional manifold S^{10}/Γ or $S^8/\Gamma \times T^2$, where Γ is chosen so that the resulting manifold is Spin^c ; for instance, take $|\Gamma|$ to be odd. Then $\hat{A}(S^8/\Gamma) \hat{A}(T^2)$ can be arranged to be 1 with an appropriate choice of Spin structure (see section 5.2 for more details on the \hat{A} -genus in these dimensions). Consider the trivial E_8 vector bundle $S^8/\Gamma \times T^2 \times \mathbb{R}^{248}$; then, using $\hat{A}(X^{10}) = 1$ and $a(E) = 0$ in this case, the expression (3.37) simplifies considerably to

$$\frac{31 \cdot 11}{2^7 \cdot 3^2 \cdot 5 \cdot 7} c_1(\pi)^5. \quad (3.38)$$

If the Rarita-Schwinger index does not contribute then this expression is nontrivial when exponentiated. Requiring the phase to be integral

$$\exp 2\pi i \left(\frac{1}{4} \cdot \frac{11 \cdot 31}{2^7 \cdot 3^2 \cdot 5 \cdot 7} c_1(\pi)^5 \right) \quad (3.39)$$

then imposes the integrality condition

$$c_1(\pi)^5 \equiv 0 \pmod{2^7 \cdot 3^2 \cdot 5 \cdot 7} \quad (3.40)$$

on the Euler class of the M-theory circle.

3.8.2 The adiabatic limit when (σ_Y^c, ω_Y) is a strictly equivariant Spin^c -Dirac structure

In this case, applying the results of [22] [23] [168] [169] gives

$$\lim_{t \rightarrow 0} \overline{\eta}(D_{\pi^* E^Y, g_Y^t}) = \left\langle \hat{A}(X) e^{c_1(\sigma_X^c)/2} \text{ch}(E) \left(\frac{e^{c_1(\pi)/2}}{2 \sinh(c_1(\pi)/2)} - \frac{1}{c_1(\pi)} \right), [X^{10}] \right\rangle \pmod{\mathbb{Z}}. \quad (3.41)$$

The expansion of the expression of the adiabatic limit (3.41) gives

$$\begin{aligned}
\lim_{t \rightarrow 0} \overline{\eta}(D_{\pi^* E^Y, g_Y^t}) &= \frac{248}{2^5 \cdot 3} \left[\frac{c_1(\sigma_X^c)^5}{2^4 \cdot 5} + \frac{c_1(\pi)c_1(\sigma_X^c)^4}{2^4 \cdot 3} - \frac{c_1(\pi)^3 c_1(\sigma_X^c)^2}{2^2 \cdot 3 \cdot 5} + \frac{c_1(\pi)^5}{3 \cdot 5 \cdot 7} \right] \\
&+ \left(\frac{60a}{2^4 \cdot 3} + \frac{31}{2 \cdot 3} \widehat{A}_4(X^{10}) \right) \left[\frac{c_1(\sigma_X^c)^3}{2} + \frac{c_1(\pi)c_1(\sigma_X^c)^2}{2^2} - \frac{c_1(\pi)^3}{3 \cdot 5} \right] \\
&+ \left(\frac{6a^2}{2^2} + 15a\widehat{A}_4(X^{10}) + 62\widehat{A}_8(X^{10}) \right) \left[c_1(\sigma_X^c) + \frac{c_1(\pi)}{2 \cdot 3} \right] \quad (3.42)
\end{aligned}$$

Special cases. 1. When the circle bundle π is trivial, $c_1(\pi) = 0$, then the expression simplifies to

$$\lim_{t \rightarrow 0} \overline{\eta}(D_{\pi^* E^Y, g_Y^t}) = \left[\frac{31}{2^6 \cdot 3 \cdot 5} c_1(\sigma_X^c)^4 + \frac{1}{2^2} \left(\frac{5a}{2} + \frac{31}{3} \widehat{A}_4 \right) c_1(\sigma_X^c)^2 + \left(\frac{3a^2}{2} + 15a\widehat{A}_4 + 62\widehat{A}_8 \right) \right] c_1(\sigma_X^c),$$

where \widehat{A}_i in the equation mean $\widehat{A}_i(X^{10})$.

2. When the Spin^c structure is trivial, i.e. $c_1(\sigma_X^c) = 0$, the right-hand side of the expression simplifies to

$$\frac{31}{2^4 \cdot 3^2 \cdot 5 \cdot 7} c_1(\pi)^5 - \frac{1}{2 \cdot 3 \cdot 5} \left(\frac{5}{2} a + \frac{31}{3} \widehat{A}_4(X^{10}) \right) c_1(\pi)^3 + \frac{1}{2 \cdot 3} \left(\frac{3a^2}{2} + 15a\widehat{A}_4(X^{10}) + 62\widehat{A}_8(X^{10}) \right) c_1(\pi) \quad (3.43)$$

Example 1. Consider the same example as in the previous section, i.e. the product E_8 bundle $S^8/\Gamma \times T^2 \times \mathbb{R}^{248}$ or $S^{10}/\Gamma \times \mathbb{R}^{248}$, then, using $\widehat{A}(X^{10}) = 1$ and $a(E) = 0$ in this case, the expression (3.43) reduces to

$$\frac{31}{2^4 \cdot 3^2 \cdot 5 \cdot 7} c_1(\pi)^5. \quad (3.44)$$

If the Rarita-Schwinger index does not contribute then this expression is nontrivial when exponentiated. Requiring the phase to be integral

$$\exp 2\pi i \left(\frac{1}{4} \cdot \frac{31}{2^4 \cdot 3^2 \cdot 5 \cdot 7} c_1(\pi)^5 \right) \quad (3.45)$$

then imposes the integrality condition

$$c_1(\pi)^5 \equiv 0 \pmod{2^2 \cdot 3^2 \cdot 5 \cdot 7} \quad (3.46)$$

on the Euler class of the M-theory circle.

Example 2. Note that in both examples we could also use $\mathbb{R}P^5 \times \mathbb{R}P^5$ or $\mathbb{R}P^9 \times \mathbb{R}P^1$. This would require G_4 to be trivial, but that is consistent with requiring the E_8 bundle to be trivial, as above.

The adiabatic limit for the Rarita-Schwinger operator: The Rarita-Schwinger operator can also be written down using the results in [168] [169] and a similar formula holds. The same is true for the boundary case of section 3.8.

4 Ten-Dimensional Spin and Spin^c Manifolds

4.1 Even dimensions: Almost complex and Kähler manifolds.

In ten (and any even real) dimensions, almost complex manifolds are Spin^c . The first obstruction to having an almost complex structure is the third integral Stiefel-Whitney class W_3 , which is exactly the only obstruction to having a Spin^c structure.

Why Spin^c structures in ten dimensions? The main point of including even-dimensional Spin^c manifolds that are not necessarily Spin is to allow for Kähler manifolds such as complex projective spaces and Fano varieties. Such compactifications appear in eleven-dimensional supergravity [139] and in M-theory [24]. In the Lorentzian case, Spin^c manifolds also allow for a Kähler factor in the decomposition of the eleven- or ten-dimensional manifold and also allows for a factor having special holonomy (see [97]). Kähler manifolds are especially interesting if we want the M-theory manifold Y^{11} to be Spin but the type IIA manifold X^{10} to be Spin^c as then we would be in the situation of nonprojectable Spin structures discussed in section 2.6. We can have a factor such as Witten spaces $M^{p,q,r}$ (see [132]) in Y^{11} with the circle bundle identified with the M-theory circle so that the base in X^{10} is a product of complex projective spaces.

Spin^c structures on complex vector bundles over X^{10} . A complex vector bundle $E_{\mathbb{C}}$ and a complex line bundle L over X^{10} determine a Spin^c structure on $E_{\mathbb{C}}$ which is unique up to homotopy. The determinant line bundle of the Spin^c structure is isomorphic to $K^* \otimes L^2$, where K^* is the anticanonical bundle of $E_{\mathbb{C}}$. Homotopic (respectively, bundle equivalent) complex structures on $E_{\mathbb{C}}$ give rise to homotopic (respectively, bundle equivalent) Spin^c structures (see e.g. Appendix D in [80]). We thus can use a complex $SU(5)$ bundle on X^{10} to define a Spin^c structure on that bundle. In fact, we can consider the decomposition $E_8 \supset (SU(5) \times SU(5))/\mathbb{Z}_5$ to define Spin^c structures on the corresponding product bundles. These encode the Ramond-Ramond fields in the reduction of M-theory to type IIA string theory [54]. Note that $SU(5)$ structures in relation to the geometry of Killing spinors in eleven dimensions are discussed in [77].

4.1.1 Kähler manifolds

Kähler manifolds are Spin^c . There is no homomorphism $U(5) \subset SO(10) \rightarrow \text{Spin}(10)$ but there is a homomorphism $U(5) \rightarrow \text{Spin}^c(10)$ that covers the inclusion $U(5) \rightarrow SO(10) \times U(1)$ which maps A to $(A, \det A)$.

Complex projective space. The complex projective space $\mathbb{C}P^m$ has a canonical Spin^c structure for all m . The twisting line bundle in this case is the canonical line bundle H , since

$$H \otimes (TCP^m \oplus 1) \cong (m+1)(H^* \otimes H) \cong m+1 \quad (4.1)$$

is trivial. Recall also the following situation in the real case. Let L be the nontrivial flat line bundle over $\mathbb{R}P^n$. Let $x = w_1(L) \in H^1(\mathbb{R}P^m; \mathbb{Z}_2)$. Then $(4k+2)L$ admits a Spin^c structure since for this line bundle $w_1 = 0$ and $w_2 = x^2$ lifts to $H^2(\mathbb{R}P^2; \mathbb{Z})$. From $T(\mathbb{R}P^m) \oplus 1 = (m+1)L$ we see that $\mathbb{R}P^{4k+1}$ admits a Spin^c structure.

Weighted complex projective space. The weighted complex projective space of dimension n is the quotient by the \mathbb{C}^* -action on the punctured complex space of dimension $n+1$,

$$\mathbb{C}P(a_0, \dots, a_n) = \frac{\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}}{\mathbb{C}^*} \quad (4.2)$$

where the action of \mathbb{C}^* on $\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}$ is given by $t \cdot (z_j) = (t^{a_j} z_j)$, with $(z_j) \in \mathbb{C}^{n+1}$. The set of all complex line bundles over $\mathbb{C}P(a_0, \dots, a_n)$ is the topological Picard group $\text{Pic}^t(\mathbb{C}P(a_0, \dots, a_n))$. This is isomorphic to \mathbb{Z} ,

$$\begin{aligned} \text{Pic}^t(\mathbb{C}P(a_0, \dots, a_n)) &\xrightarrow{\cong} \mathbb{Z} \\ [L^m] &\mapsto m, \end{aligned} \quad (4.3)$$

where L is the hyperplane line bundle. Therefore, there is a \mathbb{Z} worth of Spin^c structures on a weighted complex projective space (cf. [74]).

4.1.2 Almost complex structures

An even-dimensional manifold M^{2n} is said to have an almost complex structure (acs) if there exists a complex n -plane bundle E_{acs} on M^{2n} whose underlying real $2n$ -plane bundle is isomorphic to TM^{2n} . The manifold

M^{2n} is said to have a *stable acs* if TM^{2n} is stably isomorphic to the underlying real bundle of some complex vector bundle over M^{2n} . In [88] necessary and sufficient conditions for the existence of acs in terms of the cohomology ring and characteristic classes of M^{2n} are determined. A result of E. Thomas [158] shows that a stable almost complex structure E_{acs} on M induces an almost complex structure iff $c_n(E_{acs})$ is equal to the Euler class of M : $c_n(E_{acs}) = \chi(M^{2n})$.

The index theorem. The index theorem takes the following form. If M^{2n} is a manifold, $x \in H^2(M^{2n}; \mathbb{Z})$ a class such that the mod 2 reduction of x is $r_2(x) = w_2(M^{2n})$ and ξ a complex vector bundle over M^{2n} then

$$\langle \text{ch}(\xi) \cdot e^{x/2} \cdot \widehat{A}(M^{2n}), [M^{2n}] \rangle \quad (4.4)$$

is an integer. This is a special instance of the Spin^c index theorem for a manifold without boundary.

How many acs? Let M be a $2n$ -dimensional manifold with almost complex structure τ . The stable almost complex structures of M are given by the coset $\tau + \ker(r)$, where $r : K(M) \rightarrow KO(M)$ denotes the realification map. From [158] the almost complex structures on M are given by $\tau + F$, where $F \in \ker(r)$ satisfies $c_n(\tau + F) = c_n(\tau)$, i.e. satisfies

$$c_n(F) + c_{n-1}(F) \cdot c_1(\tau) + \cdots + c_1(F) \cdot c_{n-1}(\tau) = 0. \quad (4.5)$$

This shows that the set of almost complex structures only depends on $c(\tau)$ and the homotopy type of M [50].

1. Let M be a $2n$ -dimensional almost complex manifold. Assume there exists $x \in H^2(M; \mathbb{Q})$ such that $x^n \neq 0$ (e.g. M is a symplectic manifold). Then M admits infinitely many almost complex structures if $n \neq 1, 2, 4$ [50].
2. Let G be a compact connected simple Lie group and U a maximal closed connected subgroup of maximal rank. Among the homogeneous manifolds $M = G/U$ there are many that are almost complex.
3. Let $M = G/U$ as above. Then M admits only finitely many almost complex structures iff M is isomorphic to [50]

$$S^2, \quad S^6, \quad \mathbb{C}P^2, \quad \mathbb{C}P^4, \quad \frac{SO(6)}{SO(4) \times SO(2)}, \quad \text{or} \quad \frac{U(4)}{U(2) \times U(2)} \quad (4.6)$$

Spherical space forms. Consider a spherical space form $M(\Gamma, \varrho)$. Let $m = 2k - 1$ be odd and let $\varrho : \Gamma \rightarrow O(m)$ be fixed-point-free. Recall that ϱ is conjugate to a unitary fixed-point-free representation [167]

$$\varrho : \Gamma \rightarrow U(k) \subseteq SO(2k) \subset O(2k). \quad (4.7)$$

This shows that the stable tangent space $T(M(\Gamma, \varrho)) \oplus 1$ is the underlying real bundle of the complex vector bundle $\pi(\varrho)$ corresponding to the representation ϱ so $M(\Gamma, \varrho)$ admits a *stable complex structure* [78].

Dimension eight. A manifold M^8 has an acs iff [88]

1. $w_8(M^8) \in Sq^2 H^6(M^8; \mathbb{Z})$
and there exist cohomology classes $u \in H^2(M^8; \mathbb{Z})$ and $v \in H^6(M^8; \mathbb{Z})$ such that
2. $r_2(u) = w_2(M^8)$, $r_2(v) = w_6(M^8)$
3. $\chi(M^8) + u \cdot v \equiv 0 \pmod{4}$
4. $8\chi(M^8) = 4p_2(M^8) + 8u \cdot v - u^4 + 2u^2 \cdot p_1(M^8) - p_1(M^8)^2$.
(In case $Sq^2 H^6(M^8; \mathbb{Z}) = 0$ then the third is implied by the first and second).

Let M^8 be a manifold with a stable acs. Let (u, v) be a pair of classes in $H^2(M^8; \mathbb{Z}) \times H^6(M^8; \mathbb{Z})$ such that $r_2(u) = w_2(M^8)$ and $r_2(v) = w_6(M^8)$. Then M^8 has a stable acs E_{acs} such that $c_1(E_{acs}) = u$ and $c_3(E_{acs}) = v$ iff

$$2\chi(M^8) + u \cdot v \equiv 0 \pmod{4}. \quad (4.8)$$

Dimension ten and orientation in generalized cohomology. The existence of an almost complex structure in dimension ten is close to both the existence of Spin^c structure and the existence of an $EO(2)$ -orientation of [103]. If X^{10} is a manifold such that $H_1(X^{10}; \mathbb{Z}_2) = 0$ and $w_4(X^{10}) = 0$ then X^{10} has a stable acs iff $W_3(X^{10}) = 0$ [158]. Furthermore, if $x \in H^4(X^{10}; \mathbb{Z})$ then

$$Sq^4 x = Sq^2(x \cup w_2(X^{10})). \quad (4.9)$$

Let us consider some consequences of this for the partition function. Applying the Bockstein $\beta = Sq^1$ and using the Cartan formula in (4.9), we get (abbreviating $w_2(X^{10})$ by w_2 in the third and fourth lines)

$$\begin{aligned} \beta Sq^4 x &= Sq^3(x \cup w_2(X^{10})) \\ &= \sum_{i+j=3} Sq^i(x) \cup Sq^j(w_2(X^{10})) \\ &= Sq^0(x) \cup Sq^3(w_2) + Sq^1(x) \cup Sq^2(w_2) + Sq^2(x) \cup Sq^1(w_2) + Sq^3(x) \cup Sq^0(w_2) \\ &= x \cup 0 + Sq^1(x) \cup w_2^2 + Sq^2(x) \cup W_3 + Sq^3(x) \cup w_2. \end{aligned} \quad (4.10)$$

Now if X^{10} is Spin , i.e. if $w_2(X^{10}) = 0$, then the right-hand side is not zero in general. However, if $w_2(X^{10}) \neq 0$ but X^{10} is only Spin^c , i.e. $W_3(X^{10}) = 0$, the right hand side will be equal to $Sq^3(x) \cup w_2(X^{10})$. Take $x = r_2(G_4)$. Then if $W_7(X^{10}) = 0$ (take $Sq^3(x) = W_7$), then the right hand side is zero. Having $W_7(X^{10}) = 0$ while $w_2(X^{10}) \neq 0$ is equivalent to orientation with respect to Morava E-theory [103].

Suppose X^{10} is a manifold such that $H_1(X^{10}; \mathbb{Z}_2) = 0$ and $w_4(X^{10}) = 0$. Then if $u \in H^2(X^{10}; \mathbb{Z})$ is a class such that $r_2(u) = w_2(X^{10})$, there exists a stable acs E_{acs} on X^{10} such that $c_1(E_{acs}) = u$ and $c_3(E_{acs}) = 2u^3$. Let (u, v) be a pair of classes in $H^2(X^{10}; \mathbb{Z}) \times H^6(X^{10}; \mathbb{Z})$. Then there exists a stable acs E_{acs} on X^{10} such that $c_1(E_{acs}) = u$ and $c_3(E_{acs}) = v$ iff $r_2(u) = w_2(X^{10})$ and $v = 2u^3 + 2x$, where $Sq^2 x = 0$.

Let X^{10} be a 10-manifold with $H_1(X^{10}; \mathbb{Z}) = 0$, $H_i(X^{10}; \mathbb{Z})$ contains no torsion for $i = 2, 3$, and $H^2(X^{10}; \mathbb{Z})$ is generated by h and $h^2 \equiv 0 \pmod{2}$. Then X^{10} admits a stable acs iff [50] $w_2(X^{10}) \cdot w_4(X^{10})^2 = 0$. Thus the existence of an acs only depends on the homotopy type of X^{10} . We see from the above discussion that there are plenty of examples.

Dependence on almost complex structures and a potential anomaly. In the presence of the RR field F_2 , we have a line bundle with first Chern class e . The phase of the M-theory partition function in this case is given by [54]

$$\Omega_M(e, a) = (-1)^{f(a)} \exp \left[2\pi i \left\langle \frac{e^5}{60} + \frac{e^3 a}{6} - \frac{11e^3 \lambda}{144} - \frac{e\lambda a}{24} + \frac{e\lambda^2}{48} - \frac{e\hat{A}_8}{2}, [X^{10}] \right\rangle \right]. \quad (4.11)$$

When the characteristic classes a , λ and p_2 vanish, the expression for the phase reduces to

$$\Omega_M(e, 0) = \exp \left[2\pi i \left\langle \frac{e^5}{60}, [X^{10}] \right\rangle \right] \quad (4.12)$$

Take $e = c_1(X^{10})$. There are examples of two 5-dimensional algebraic varieties which are C^∞ -diffeomorphic, but have different Chern numbers [32], namely the ten-dimensional flag manifold

$$X^{10} = U(4)/U(2) \times U(1) \times U(1), \quad (4.13)$$

whose points are the ordered triples of one 2-dimensional and two 1-dimensional linear subspaces of the standard hermitian spaces \mathbb{C}^4 which are pairwise Hermitian orthogonal. The manifold X^{10} carries two homogeneous complex structures:

1. the 5-dimensional complex manifold X_1^{10} consisting of the flags in \mathbb{C}^4 of type $(0) \subset (1) \subset (3) \subset (4)$, i.e. the origin is contained in a one-dimensional linear subspace, contained in a three-dimensional linear subspace, contained in \mathbb{C}^4 .
2. the 5-dimensional complex manifold X_2^{10} consisting of flags of type $(0) \subset (1) \subset (2) \subset (4)$.

In [32] it was shown that the first Chern classes and the Chern numbers of X_1^{10} and X_2^{10} are different: $c_1(X_1^{10})$ is divisible by 3 while $c_1(X_2^{10})$ is not divisible by 3, and the particular Chern numbers are

$$c_1^5[X_1^{10}] = 2^2 \cdot 3^5 \cdot 5 = 4860, \quad (4.14)$$

$$c_1^5[X_2^{10}] = 2^2 \cdot 3^3 \cdot 5^3 = 4500. \quad (4.15)$$

Metrics g_1 and g_2 on X_1^{10} and X_2^{10} , respectively, are equivalent when the two manifolds are considered as differentiable manifolds but *not* when the manifolds are considered as complex analytic manifolds. This implies that if we consider a family of metrics $g_t = tg_1 + (1-t)g_0$ through which a coordinate transformation takes g_0 to g_1 does not necessarily leave invariant the phase of the partition function (4.12); instead the change is given by $\exp 2\pi i \Delta S$, where $\Delta S = 360$. However, in the particular example, the effective action (or phase) is invariant. Of course, one might be able to come up with other examples where the action does not change by a multiple of 60 and then the phase would pick up a potential anomaly. This provides a concrete physical realization of speculations given in [129] on the hope for a role of such different values of Chern numbers in anomalies.

4.2 Using the M-theory circle to define a Spin^c square root

Let X^{10} be an almost complex manifold and let $K_X = \Lambda^{5,0}(X^{10})$ be the canonical bundle of X^{10} , where $\Lambda^{r,0}$ denotes the bundle of r -forms of holomorphic type. The canonical line bundle does not admit a square root as the space X^{10} is not necessarily Spin. However, this bundle admits a square root when tensored with an appropriate power of the complex line bundle \mathcal{L} . We will take this line bundle to be associated to the M-theory circle bundle. Hence we will use the M-theory circle to define spinors appropriately. Note that line bundles related to other entities can be used to define Spin^c structures, e.g. for the B -field B_2 introduced in the context of D-branes [70]. Note also that in A - and B -twisted topological field theories [164] the nature of spinors is changed by tensoring the spin bundle (which is the square root of the canonical class in this case) with a line bundle.

In what follows we make use of some constructions from [90]. Isomorphism classes of complex line bundles or S^1 -principal bundles on X^{10} are parametrized by the isomorphism of cohomology groups

$$\begin{aligned} c_1 : H^1(X^{10}, \underline{U(1)}) &\cong H^2(X^{10}; \mathbb{Z}) \\ \underline{L} &\longmapsto c_1(L). \end{aligned} \quad (4.16)$$

where $\underline{U(1)}$ is the sheaf of germs of circle-valued functions. Suppose that $c_1(K_X)$ is a nonzero cohomology class. Let $p \in \mathbb{N}$ be the greatest number such that $c = \frac{1}{p}c_1(K_X) \in H^2(X^{10}; \mathbb{Z})$. When X^{10} is Kähler, p is called the *index* of the manifold, and it is called the *minimal Chern number* of X^{10} when the latter is symplectic. If we take the M-theory circle bundle \mathcal{L} such that $c_1(\mathcal{L}) = c = \frac{1}{p}c_1(K_X)$ then $K_X = \mathcal{L}^p$, i.e. \mathcal{L} is the p -th root of the canonical bundle K_X , and so \mathcal{L} is the p -fold covering space of K_X .

Example: The index of $\mathbb{C}P^5$ with the usual complex structure is 6. The only line bundle \mathcal{L} with first Chern class $\frac{1}{6}c_1(K_{\mathbb{C}P^5})$ is the universal bundle over $\mathbb{C}P^5$ [82] and the corresponding principal S^1 -bundle \mathcal{L} is the Hopf bundle $S^{11} \rightarrow \mathbb{C}P^5$. Then $K_{\mathbb{C}P^5} = \mathcal{L}^6 = S^{11}/\mathbb{Z}_6$ so that $(K_{\mathbb{C}P^5})^2 = \mathcal{L}^{12} = S^{11}/\mathbb{Z}_{12}$. We could also

deduce that $c_1(K_{\mathbb{C}P^5})$ is divisible by 3 from the fact that $\mathbb{C}P^5$ is a complex contact manifold, i.e. it carries a codimension one holomorphic subbundle of $T^{1,0}\mathbb{C}P^5$ which is maximally non-integrable.

Spin and Spin^c structures on Y^{11} from Spin^c structures on X^{10} . Next we consider the situation in eleven dimensions. In going from ten to eleven dimensions, we could consider pullback of structures. Given a Spin^c structure on X^{10} with determinant line bundle $\mathbb{L} \in H^2(X^{10}; \mathbb{Z})$ the oriented Riemannian submersion $\pi : \mathcal{L} \rightarrow X^{10}$, associated to $\pi : Y^{11} \rightarrow X^{10}$, induces a Spin^c structure on \mathcal{L} whose determinant line bundle is $\pi^*(\mathbb{L}) \in H^2(\mathcal{L}; \mathbb{Z})$. Now, from the Gysin sequence for the circle bundle

$$0 \longrightarrow H^0(X^{10}; \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\cup c_1(\mathcal{L})=\alpha} H^2(X^{10}; \mathbb{Z}) \xrightarrow{\pi^*} H^2(\mathcal{L}; \mathbb{Z}) \longrightarrow 0 \quad (4.17)$$

we have that $\ker \pi^* = \mathbb{Z}(\alpha)$. This implies the following.

1. Spin^c structures on X^{10} whose determinant line bundles are tensor powers \mathcal{L}^q , for $q \in \mathbb{Z}$, are exactly those inducing on \mathcal{L} , through the projection π , the unique Spin structure.
2. The ones that induce a Spin^c structure, however, are the ones for which $c_1(\mathcal{L}^q) \equiv c_1(X^{10}) \pmod{2}$.

Examples. Now we specialize to Kähler manifolds. Using $c_1(\mathcal{L}) = \alpha$ and $c_1(X^{10}) = -c_1(K_X) = -p\alpha$, with p the index of X^{10} , we get

$$(q + p)c \equiv 0 \pmod{2} \in H^2(X^{10}; \mathbb{Z}), \quad (4.18)$$

which in turn means that $p + q \in 2\mathbb{Z}$ since c is an indivisible class in $H^2(X^{10}; \mathbb{Z})$. For example if $X^{10} = \mathbb{C}P^5$ then $p = 6$ is already even and so in order for $\mathcal{L}^q \otimes K_{\mathbb{C}P^5}$ to have a square root, q has to be even. That is, we are considering only even Kaluza-Klein modes in the Fourier mode spectrum. The argument for the other (perhaps more phenomenologically realistic) Fano varieties is similar.

Dimensional reduction of \mathcal{L}^q . Now let us consider the odd case, i.e. Y^{11} is Spin and X^{10} is Spin^c . Let us also specialize to the class of examples where X^{10} is Kähler. Then we have a Kähler form Ω^X on X^{10} . We would like to consider how spinors and spinor bundles on X^{10} and \mathcal{L}^q are related.

The Spin bundles: Denote by $S^q X^{10}$ the spinor bundle of the corresponding Spin^c structure with determinant line bundle \mathcal{L}^q with $q \in -p + 2\mathbb{Z}$. All these bundles pull back to the spinor bundle $S\mathcal{L}$,

$$\pi^*(S^q X^{10}) = S\mathcal{L} \quad \forall q \in \mathbb{Z}. \quad (4.19)$$

The Levi-Civita connections: The Levi-Civita connection ∇^X on X^{10} gives a corresponding S^1 -connection on the principal S^1 bundle $\mathcal{L} \rightarrow X^{10}$ given by a 1-form $i\theta \in \Gamma(\Lambda^1(\mathcal{L}) \otimes i\mathbb{R})$. The metrics are related, via π and θ , as $g_{\mathcal{L}} = \pi^*(g_X) + \frac{p^2}{36}\theta \otimes \theta$ such that the vector field v defined by the free circle action on \mathcal{L} has constant length $p/6$, i.e. the fibers are totally geodesic circles of length $p\pi/3$. Note that the above quantities have normalized values for $\mathbb{C}P^5$, which has index $p = 6$. For each integer $q \in \mathbb{Z}$, there is a Levi-Civita connection on the line bundle \mathcal{L}^q given by $i\theta^q \in \Gamma(\Lambda^1(\mathcal{L}^q) \otimes i\mathbb{R})$, which is mapped via the pullbacks of the q -fold covering $pr_{1,q} : \mathcal{L} \rightarrow \mathcal{L}^q$ as $pr_{1,q}^* \theta^q = q\theta$. Hence the metric on \mathcal{L}^q is

$$g_{\mathcal{L}^q} = \pi^*(g_X) + \frac{q^2 p^2}{36} \theta \otimes \theta \quad (4.20)$$

Obviously, the larger q is the larger is the part of the metric coming from the circle part. This limit is in a sense the inverse of that of the adiabatic limit, for which the volume of the base is taken to be very large compared to that of the fiber (cf. section 7.3).

Correspondence with Spin structures. A Spin^c structure with trivial canonical line bundle is canonically identified with a Spin structure. Since the corresponding $U(1)$ bundle is trivial then it admits a section s . Then the inverse image $P_{\text{Spin}(11)}(Y^{11})$ of $P_{SO(11)}Y^{11} \times s$ defines the desired Spin structure on Y^{11} . Furthermore, if the connection ∇^L of L is flat, then the Spin^c connection corresponds to the connection on the Spinor bundle. Thus, in using the M-theory circle to define the canonical line bundle, if that circle bundle admits a flat connection then we have a Spin structure.

4.3 Spin^c ten-dimensional manifolds with G -actions

Let X^{10} be a Spin^c -manifold on which a compact (not necessarily connected) Lie group G acts, and let S^1 denote a fixed subgroup of G . Let V (respectively W) be a complex (respectively Spin) vector bundle over X^{10} . The question is whether the S^1 -action lifts to the Spin^c structure and the vector bundles V and W . Let $X_G := EG \times_G X$ denote the Borel construction, where EG is the classifying space for G . Let L be a complex line bundle over X . Then the G -action lifts to L iff $c_1(L)$ is in the image of the forgetful homomorphism $H^2(X_G; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})$ [86].

Circle action. Let Q_c denote the Spin^c structure on X^{10} . This induces two complex line bundles:

1. L_c , a complex line bundle over X^{10} defined by the $U(1)$ principal bundle $Q_c/\text{Spin}(10) \rightarrow Q_c/\text{Spin}^c(10) \cong X^{10}$ using the standard embedding of $\text{Spin}(10)$ in $\text{Spin}^c(10)$. The class $c_1(L_c)$, also denoted $c_1(Q_c)$ or $c_1(X^{10})$, will be called the first Chern class of X^{10} .
2. The group $U(1)$ acts on Q_c via the embedding $U(1) \hookrightarrow \text{Spin}^c(10)$. The quotient $Q_c/U(1)$ is the $SO(10)$ principal bundle P of orthonormal frames, for the metric induced by Q_c . The projection $\xi : Q_c \rightarrow P$ is a $U(1)$ principal bundle and defines a second complex line bundle ξ .

The two line bundles are related: The pullback of L_c to P is isomorphic to ξ^2 .

We now consider the possible lift of the circle action. The S^1 action on X^{10} lifts to P via differentials. If the S^1 -action lifts to the principal $U(1)$ bundle $\xi : Q_c \rightarrow P$ then for a modified lift the Spin^c structure $Q_c \rightarrow X$ is S^1 -equivariant [136]. Furthermore, If the first Betti number $b_1(X)$ vanishes or $c_1(X)$ is a torsion element then the S^1 -action lifts to the Spin^c structure Q_c . This can be shown as follows [51]. The S^1 -action on P lifts to the principal $U(1)$ bundle $\xi : Q_c \rightarrow P$ iff $c_1(\xi)$ is in the image of $H^2(P_{S^1}; \mathbb{Z}) \rightarrow H^2(P; \mathbb{Z})$. Consider the Leray-Serre spectral sequence $\{E_r^{p,q}\}$ for $P_{S^1} \rightarrow BS^1$ in integral cohomology. Since $H^*(BS^1; \mathbb{Z})$ is a polynomial ring in one generator of degree 2, the only possibly nontrivial differential in the spectral sequence restricted to the subgroup of bidegree $(0, 2)$ is

$$d_2 : E_2^{0,2} \rightarrow E_2^{2,1} \cong H^2(BS^1; H^1(P; \mathbb{Z})) . \quad (4.21)$$

The Leray-Serre spectral sequence for $P \rightarrow X$ shows that $b_1(P)$ vanishes if $b_1(X)$ does. In this case, $H^2(BS^1; H^1(P; \mathbb{Z})) = 0$ and d_2 is the zero map.

If $c_1(X)$ is a torsion class then $c_1(\xi)$ is also torsion since the pullback of L_c to P is isomorphic to ξ^2 . Since $E_2^{2,1} \cong H^2(BS^1; H^1(P; \mathbb{Z}))$ is always torsion-free, the image of $c_1(\xi)$ under d_2 is zero. Therefore, the class $c_1(\xi)$ survives and all differentials vanish on it, implying that it lies in the image of $H^2(P_{S^1}; \mathbb{Z}) \rightarrow H^2(P; \mathbb{Z})$. Thus, the S^1 -action on P admits a lift to Q_c for which the Spin^c structure $Q_c \rightarrow X$ is equivariant.

4.4 Category of Spin representations and the Atiyah-Bott-Shapiro construction

The Atiyah-Bott-Shapiro (ABS) construction [9] relates the Grothendieck groups of real Clifford algebras to the KO-theory of spheres. Consider the following algebraic objects over a manifold X :

◦ $V_{\mathbb{R}}(X)$, the set of isomorphism classes of real vector bundles over X . It is an abelian semigroup with the addition operation being the Whitney (direct) sum;

$\circ F_{\mathbb{R}}(X)$, the free abelian group generated by elements of $V_{\mathbb{R}}(X)$;
 $\circ E_{\mathbb{R}}(X)$, the subgroup of $F_{\mathbb{R}}(X)$ generated by elements of the form $[V] + [W] - ([V] \oplus [W])$, where $+$ denotes addition in $F_{\mathbb{R}}(X)$ and \oplus denotes addition in $V_{\mathbb{R}}(X)$. Then the real K-theory of X is defined to be the abelian group $KO(X) = F_{\mathbb{R}}(X)/E_{\mathbb{R}}(X)$ whose elements are virtual bundles.

The tensor product of two Clifford algebras $C\ell_n \otimes C\ell_m$ is in general not a Clifford algebra. Thus, to find a multiplicative structure in the representations of Clifford algebras it is natural to consider the category of \mathbb{Z}_2 -graded modules [107]. A \mathbb{Z}_2 -graded module for $C\ell_{11}$ is a module W with a decomposition $W = W^0 \oplus W^1$ such that

$$\begin{aligned} C\ell_{11}^0 \cdot W^0 &\subseteq W^0, & C\ell_{11}^0 \cdot W^1 &\subseteq W^1, \\ C\ell_{11}^1 \cdot W^0 &\subseteq W^1, & C\ell_{11}^1 \cdot W^1 &\subseteq W^0. \end{aligned} \quad (4.22)$$

The category of \mathbb{Z}_2 -graded modules over $C\ell_{11}$ is equivalent to the category of ungraded modules over $C\ell_{10} \cong C\ell_{11}^0$ by passing from the graded module $W^0 \oplus W^1$ to the module W^0 . Define \mathcal{M}_n to be the Grothendieck group of equivalence classes of irreducible representations of $C\ell_n$ and let $\widehat{\mathcal{M}}_n$ be the Grothendieck group of real \mathbb{Z}_2 -graded modules over $C\ell_n$. Thus, there is an isomorphism $\widehat{\mathcal{M}}_{11} = \mathcal{M}_{10}$. (See the end of the third paragraph in section 2.5 for an instance of this).

A natural \mathbb{Z}_2 -graded tensor product of \mathbb{Z}_2 -graded modules $W = W^0 \oplus W^1$ and $V = V^0 \oplus V^1$ over $C\ell_n$ and $C\ell_m$ respectively, is defined as follows. Set

$$\begin{aligned} (W \hat{\otimes} V)^0 &= W^0 \otimes V^0 + W^1 \otimes V^1 \\ (W \hat{\otimes} V)^1 &= W^0 \otimes V^1 + W^1 \otimes V^0. \end{aligned} \quad (4.23)$$

The action of $C\ell_n \hat{\otimes} C\ell_m$ on $W \hat{\otimes} V$ is given by $(\varphi \otimes \psi) \cdot (w \otimes v) = (-1)^{pq}(\varphi w) \otimes (\psi v)$, where $\deg(\psi) = p$ and $\deg(w) = q$. The \mathbb{Z}_2 -graded tensor product induces a natural associative pairing $\widehat{\mathcal{M}}_n \otimes_{\mathbb{Z}} \widehat{\mathcal{M}}_m \rightarrow \widehat{\mathcal{M}}_{n+m}$ which gives $\widehat{\mathcal{M}}_* = \bigoplus_{n \geq 1} \widehat{\mathcal{M}}_n$ the structure of a graded ring. This multiplication gives

$$\left(\widehat{\mathcal{M}}_*/i^* \widehat{\mathcal{M}}_{*+1} \right) \equiv \bigoplus_{n \geq 0} \left(\widehat{\mathcal{M}}_n/i^* \widehat{\mathcal{M}}_{n+1} \right). \quad (4.24)$$

Let $W = W^0 \oplus W^1$ be a \mathbb{Z}_2 -graded module over the Clifford algebra $C\ell_{10} \equiv C\ell(\mathbb{R}^{10})$. Let $\mathbb{D}^{10} = \{x \in \mathbb{R}^{10} : \|x\| \leq 1\}$ be the unit disk and set $S^9 = \partial \mathbb{D}^{10}$. Let $E_0 = \mathbb{D}^{10} \times W^0$ and $E_1 = \mathbb{D}^{10} \times W^1$ be the trivial product bundles and let $\mu : E_0 \xrightarrow{\sim} E_1$ be the isomorphism over S^9 given by Clifford multiplication $\mu(x, w) \equiv (x, x \cdot w)$. Associate to the graded module W the element $\varphi(W) = [E_0, E_1; \mu] \in K(\mathbb{D}^{10}, S^9)$, which depends only on the isomorphism class of the graded module W , with the map $W \mapsto \varphi(W)$ being an additive homomorphism. Thus, this gives a homomorphism

$$\varphi : \widehat{\mathcal{M}}_{11} \rightarrow KO(\mathbb{D}^{10}, S^9) = \widetilde{KO}(S^{10}) = KO^{-10}(\text{pt}). \quad (4.25)$$

Then there is a graded ABS isomorphism

$$\varphi_* : \left(\widehat{\mathcal{M}}_*/i^* \widehat{\mathcal{M}}_{*+1} \right) \rightarrow KO^{-*}(\text{pt}), \quad (4.26)$$

giving a Clifford algebra definition of the α -invariant. Further definitions and discussions are given in section 5.2.

The ABS isomorphism provides explicit generators for $KO^{-*}(\text{pt})$ defined via representations of Clifford algebras. Let $S = S^+ \oplus S^-$ be the fundamental graded module for $C\ell_{4n}$ where $S^{\pm} = (1 \pm \omega)S$, where ω is the volume element. Then $\sigma_{4n} \equiv [S^+, S^-; \mu]$ is a generator of the group $KO^{-4n}(\text{pt}) \cong KO_{\text{cpt}}(\mathbb{R}^{4n}) \cong \mathbb{Z}$, where $\mu : S^+ \rightarrow S^-$ denotes Clifford multiplication by $x \in \mathbb{R}^{4n}$. Clifford algebra relations give the relation $4\sigma_8 = (\sigma_4)^2$.

⁵e.g. $S^+ \oplus S^-$ to S^+ .

The Clifford linear Dirac operator. We utilize a unified approach which involves general Clifford modules, i.e. takes into account all possible vector bundles coming from a given principal Spin bundle. The starting point is a principal Spin bundle $P_{\text{Spin}}(X)$. Then in order to do analysis we need an associated vector bundle. This can be associated to one of the spinor representations $\Delta, \Delta^+, \Delta^-$, etc., depending on the dimension. However, there is a formulation via Clifford algebras which automatically takes care of all cases without having to worry about making such (dimension-dependent) choices. Furthermore, it allows for generalization to coupling to other vector bundles. This is the concept of a Clifford-linear Dirac operator (see [107]) associated to the bundle

$$\mathcal{S} = P_{\text{Spin}}(X) \times_{\text{Spin}(10)} C\ell_{10} . \quad (4.27)$$

In constructing the bundle we use the left action of the Clifford algebra and we can make use of the right action to define the Dirac operator. This makes $\ker D$ not just a vector space but a \mathbb{Z}_2 -graded Clifford module.

Over a compact 10-manifold X any $C\ell_{10}$ -linear Dirac operator D has an analytic index $\text{Ind}_{10}(D) \in KO^{-10}(\text{pt})$ defined by applying the Atiyah-Bott-Shapiro isomorphism to the residue class of the Clifford module $[\ker D]$ as follows (cf. [107]). Consider a Clifford bundle $C\ell(X)$, which is the bundle of Clifford algebras over X whose fiber at $x \in X$ is the Clifford algebra $C\ell(T_x^*X)$ of the Euclidean spaces T_x^*X . Define a canonical bundle map $\lambda_\omega : C\ell(X) \rightarrow C\ell(X)$ by setting $\lambda_\omega(\varphi) = \omega \cdot \varphi$. In ten dimensions this satisfies $\lambda_\omega^2 = -1$, so that if we complexify S and consider the operator $i\lambda_\omega$ we get a splitting $S \otimes \mathbb{C} = (S \otimes \mathbb{C})^+ \oplus (S \otimes \mathbb{C})^-$, i.e. λ_ω defines a complex structure on S with the above decomposition corresponding to the $(1,0), (0,1)$ decomposition for the complex structure. Let $\mathcal{S} = \mathcal{S}^0 \oplus \mathcal{S}^1$ be a \mathbb{Z}_2 -graded $C\ell_{10}$ -Dirac bundle over a compact 10-manifold X . Then the analytic index of the Dirac operator \mathcal{D} of \mathcal{S} is the residue class

$$[\ker \mathcal{D}] \in \widehat{\mathcal{M}}_{10}/i^*\widehat{\mathcal{M}}_{11} \cong KO^{-10}(\text{pt}) = \mathbb{Z}_2 , \quad (4.28)$$

or, equivalently, via $\mathcal{D}^0 : \Gamma(\mathcal{S}^0) \rightarrow \Gamma(\mathcal{S}^1)$,

$$[\ker \mathcal{D}^0] \in \mathcal{M}_0/i^*\mathcal{M}_{10} \cong KO^{-10}(\text{pt}) = \mathbb{Z}_2 . \quad (4.29)$$

Examples. Let $S = S^+ \oplus S^-$ be the ordinary complex \mathbb{Z}_2 -graded Dirac bundle over a compact ten-dimensional manifold X .

- First, take V to be the complexified tangent bundle $T_{\mathbb{C}}X$, which splits into the \mathbb{Z}_2 -graded decomposition $T_{\mathbb{C}}X = T \oplus \overline{T}$, where T is the holomorphic and \overline{T} is the antiholomorphic tangent bundle. Then a \mathbb{Z}_2 -graded $C\ell_{10}$ -bundle \mathcal{S} is

$$\begin{aligned} \mathcal{S} = S \otimes T_{\mathbb{C}}X &= (S^+ \oplus S^-) \otimes (T \oplus \overline{T}) \\ &= (S^+ \otimes T \oplus S^- \otimes \overline{T}) \oplus (S^+ \otimes \overline{T} \oplus S^- \otimes T) \\ &:= \mathcal{S}^+ \oplus \mathcal{S}^- . \end{aligned} \quad (4.30)$$

- Next, take V to be an E_8 bundle. The inclusion $(SU(5) \times SU(5))/\mathbb{Z}_5 \subset E_8$ allows us to break V into a pair of $SU(5)$ bundles as in [54], $V \supset E \oplus \overline{E}$. Then, as in the first case, we get the \mathbb{Z}_2 -graded $C\ell_{10}$ -module

$$\begin{aligned} \mathcal{S} = S \otimes V &= (S^+ \oplus S^-) \otimes (E \oplus \overline{E}) \\ &= (S^+ \otimes E \oplus S^- \otimes \overline{E}) \oplus (S^+ \otimes \overline{E} \oplus S^- \otimes E) \\ &:= \mathcal{S}^+ \oplus \mathcal{S}^- . \end{aligned} \quad (4.31)$$

Topological definition of the mod 2 index. We follow [107]. Consider the graded ring structure of $KO^*(\text{pt})$, generated over \mathbb{Z} by elements η , ω , μ of degrees 1, 4, and 8, respectively, with relations

$$2\eta = \eta\omega = \eta^3 = 0, \quad \omega^2 = 4\mu. \quad (4.32)$$

That is,

$$\begin{aligned} KO^{-1}(\text{pt}) &= \mathbb{Z}_2\eta, & KO^{-4}(\text{pt}) &= \mathbb{Z}\omega, \\ KO^{-2}(\text{pt}) &= \mathbb{Z}_2\eta^2, & KO^{-8}(\text{pt}) &= \mathbb{Z}\mu, \end{aligned} \quad (4.33)$$

with the periodicity $KO^{\pm 8}(\text{pt}) \cong KO^*(\text{pt})$. Consider the embedding $i : X^{10} \rightarrow S^{10+8k}$, where k is large, and let NX^{10} be the normal bundle of X^{10} inside S^{10+8k} . Let $U(NX^{10})$ be the KO -Thom class of NX^{10} , and let j be the natural isomorphism $\mathbb{D}N/SN \simeq S^{10+8k}$, with $\mathbb{D}N$ and SN the disk and the sphere bundle of the normal bundle, respectively. Then, for any $E \in KO(X^{10})$, define the Gysin morphism

$$f_! : KO(X^{10}) \rightarrow KO^{-10}(\text{pt}) \quad (4.34)$$

defined as

$$f_!(E) = j^*(E \cup U(NX^{10})) \in \widetilde{KO}(S^{10+8k}) \cong KO^{-10}(\text{pt}), \quad (4.35)$$

which is independent of the embedding i . The mod 2 index of the Dirac operator is then defined as [15]

$$\text{Ind}_2(D \otimes E) \equiv \pi(f_!E) \pmod{2}, \quad (4.36)$$

where $\pi : KO^{-1}(\text{pt}) \xrightarrow{\cong} \mathbb{Z}_2$, $KO^{-2}(\text{pt}) \xrightarrow{\cong} \mathbb{Z}_2$ are the isomorphisms sending η and η^2 to $1 \in \mathbb{Z}_2$.

Consider the classical genera

$$\widehat{U} = \begin{cases} \text{Ind}_2 D, & X \text{ has dimension } 1, 2, 9, \text{ or } 10 \\ \text{Ind} D, & X \text{ has dimension } 4 \text{ or } 8. \end{cases} \quad (4.37)$$

The mod 2 index on X^{10} can be related to the index on the twelve-manifold $\mathcal{T}(X^{10})$, the mapping torus of X^{10} , via the theorem proved in [112]:

$$\text{Ind}_2 D \equiv \frac{1}{2} \int_{\mathcal{T}(X)} \widehat{U}(\mathcal{T}(X)) \pmod{2}. \quad (4.38)$$

When is the mod 2 index nonzero. The mod 2 invariant given by the index map

$$\text{Ind}_2 : \Omega_{10}^{\text{Spin}} \rightarrow KO^{-10}(\text{pt}) \quad (4.39)$$

is a ring homomorphism. The nonzero element $\eta \in KO^{-1}(\text{pt})$ has the property that ηx and $\eta^2 x$ are nonzero whenever x is an odd multiple of the generator in degree 8 [107].

Example 1: the circle S^1 . Consider the nontrivial Spin structure on S^1 . The \mathbb{Z}_2 -grading on $\mathcal{C}\ell_1 \cong \mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, $\mathcal{C}\ell_1^0 \cong \mathbb{R}$, gives the corresponding grading on the Clifford module $\mathcal{S} = \mathcal{S}^0 \oplus \mathcal{S}^1 = (S^1 \times \mathbb{R}) \oplus (S^1 \times i\mathbb{R})$. Sections of \mathcal{S} are complex-valued functions on the circle. The Dirac operator $\mathcal{D} = i \frac{d}{dt}$ splits into \mathcal{D}^0 and \mathcal{D}^1 , and the kernel of $\mathcal{D} : \Gamma(\mathcal{S}^0) \rightarrow \Gamma(\mathcal{S}^1)$ is the set of real-valued constant functions on S^1 , a space whose real dimension is 1. Thus, $\text{ind}_2(S^1) \neq 0$ is the generator of $KO^{-1}(\text{pt}) \cong \mathbb{Z}_2$. On the other hand, $\text{Ind}_2 = 0$ for the trivial Spin structure on S^1 .

Example 2 : the torus $T^2 = S^1 \times S^1$. Choose the Spin structure on T^2 which is given by squaring the nontrivial Spin structure on S^1 . This is the covering

$$P_{\text{Spin}}(T^1) = T^2 \times S^1 \xrightarrow{\text{Id} \times z^2} T^2 \times S^1 = P_{SO}(T^2). \quad (4.40)$$

The Clifford algebra $\mathcal{C}\ell_2 \cong \mathbb{H}$ and $\mathcal{C}\ell_2^0 \cong \mathbb{C}$ give that $\mathcal{S}(T^2) = T^2 \times \mathbb{H}$ and $\mathcal{S}^0(T^2) = T^2 \times \mathbb{C}$. The kernel of \mathcal{D}^0 is the complex-valued constant functions. Then

$$\text{Ind}_2(T^2) \neq 0 \quad \text{for nontrivial Spin structure on } S^1. \quad (4.41)$$

Multiplicative properties of the index. Let X_1 and X_2 be two Riemannian Spin manifolds of dimensions n_1 and n_2 , with $n_1 + n_2 = 10$. Let $\mathcal{D}_k : \Gamma(\mathcal{S}_k) \rightarrow \Gamma(\mathcal{S}_k)$ be the Atiyah-Singer operator for the $\mathcal{C}\ell_{n_k}$ -Dirac bundle $\mathcal{S}_k = \mathcal{S}(X_k)$, for $k = 1, 2$. Let $\mathcal{D} : \Gamma(\mathcal{S}) \rightarrow \Gamma(\mathcal{S})$ be the operator for $X^{10} = X_1 \times X_2$ with the product Riemannian and Spin structure. For this structure, $P_{\text{Spin}}(X_1 \times X_2) \supset P_{\text{Spin}}(X_1) \times_{\mathbb{Z}_2} P_{\text{Spin}}(X_2)$, where \mathbb{Z}_2 acts on the two factors by $(-1, 1)$. Then

$$\mathcal{S} = \mathcal{S}_1 \hat{\otimes} \mathcal{S}_2, \quad (4.42)$$

the exterior \mathbb{Z}_2 -graded tensor product, which is a $\mathcal{C}\ell_{n_1+n_2} = (\mathcal{C}\ell_{n_1} \hat{\otimes} \mathcal{C}\ell_{n_2})$ -Dirac bundle. This gives [107] $\ker(\mathcal{D}) = \ker(\mathcal{D}_1) \hat{\otimes} \ker(\mathcal{D}_2)$, with a \mathbb{Z}_2 -graded tensor product inherited from the \mathbb{Z}_2 -grading of the Clifford modules (4.42) and is the multiplication in $KO^{-*} = \widehat{\mathcal{M}}_*/i^*\widehat{\mathcal{M}}_{*+1}$. Then Ind_2^* is a ring homomorphism

$$\text{Ind}_2^{n_1+n_2}(\mathcal{D}) = \text{Ind}_2^{n_1}(\mathcal{D}_1) \text{Ind}_2^{n_2}(\mathcal{D}_2). \quad (4.43)$$

Extension to the case of coupling to a vector bundle. Let E be a real vector bundle over X with an orthogonal connection. The bundle $\mathcal{S}(X) \otimes E$ is naturally a \mathbb{Z}_2 -graded $\mathcal{C}\ell_{10}$ -Dirac bundle with a Dirac operator \mathcal{D}_E . Take the Spin bundle SX and tensor it with E to get $SX \otimes E$ with Dirac operator D_E . Let $h_E(X) \equiv \ker(D_E)$ denote the space of real harmonic E -valued spinors. Then the above result on multiplicativity (4.43) extends to this case of twisted Dirac operators [107].

Note that there is also a Clifford-module approach to twisted (complex) K-theory, which is explained for instance in [99].

5 Anomalies in the Partition Function and Geometry

5.1 Effect of scalar curvature

Scalar curvature for products and bundles. Let M and N be closed Riemannian manifolds with scalar curvatures \mathcal{R}_M and \mathcal{R}_N , respectively. Let $E := M \times N$ and let $\pi(x, y) = y$ define a trivial Riemannian submersion from E to N . Consider the canonical variation given by $g_t := tds_M^2 + ds_N^2$ on E so that the scalar curvature corresponding to g_t is $\mathcal{R}_t = t^{-1}\mathcal{R}_M + \mathcal{R}_N + O(t)$. Assuming \mathcal{R}_M is positive, \mathcal{R}_t tends to ∞ as t goes to zero. Thus, if M admits a metric of positive scalar curvature then so does $M \times N$ (see e.g. [79]).

Next let $\pi : Y \rightarrow X$ be a Riemannian submersion with totally geodesic fibers. For any t , $\pi : (Y, g_Y) \rightarrow (X, g_X)$ is a Riemannian submersion with totally geodesic fibers. Let \mathcal{R}_t be the scalar curvature of the metric g_t , let \mathcal{R}_F be the scalar curvature of the original metric on the fiber F , and let \mathcal{R}_X be the scalar curvature of the metric on X . Then $\mathcal{R}_t = t^{-1}\mathcal{R}_F + \mathcal{R}_X + O(t)$. This is proved using O'Neill's formula [28]. Then a similar conclusion as for the case of products follows.

Scalar curvature vs. Spin. Scalar curvature is closely related to (non)existence of Spin structure as well as to Killing spinors. Let M be a closed simply connected manifold of dimension at least 5. If M does not admit a Spin structure, then M carries a metric of positive scalar curvature [84]. Let (M, g) be an n -dimensional connected complete Riemannian Spin manifold with a nontrivial Killing spinor with $\alpha \neq 0$. Then (see [73] [21])

1. (M, g) is locally irreducible.
2. M is Einstein with Einstein constant $\lambda = 4(n-1)\beta^2$, i.e. M is a space of constant sectional curvature equal to $4\beta^2$. In particular, when β is a nonzero real number, M is compact of positive scalar curvature.

G-action and scalar curvature. Let Y^{11} be a closed manifold equipped with an S^1 action. Then, from [25], the following are equivalent:

1. Y^{11} admits an S^1 -invariant metric of positive scalar curvature.

2. Y^{11}/S^1 admits a metric of positive scalar curvature.

The above equivalence is not true for S^1 replaced by a connected nonabelian group G . For example, consider $Y^{11} = G \times T^n$, with $G = SU(2) = Sp(1)$, $SO(3)$, $SO(4)$, $SU(3)$, $SO(5) = Sp(2)$, and n is respectively, 8, 8, 5, 3, and 1. The bi-invariant Riemannian metric on G has positive scalar metric, and G acts freely on the first factor in Y^{11} . The quotient T^n obviously does not admit a metric of positive scalar curvature.

Let Y^{11} be a closed oriented free S^1 -manifold which is simply connected and does not admit a Spin structure. Then Y^{11} carries an S^1 -invariant metric of positive scalar curvature. This can be proved as follows [85]. The long exact sequence on homotopy of the S^1 -fibration $S^1 \hookrightarrow Y^{11} \xrightarrow{\pi} X^{10} = Y^{11}/S^1$ shows that X^{10} is also simply connected. Furthermore, $TY^{11} \cong \pi^*(TX^{10} \oplus \mathcal{O}_{\mathbb{R}})$, where $\mathcal{O}_{\mathbb{R}}$ is a trivial real line bundle. Calculating w_2 , and using the fact that $w_2(\mathcal{O}_{\mathbb{R}}) = 0$ gives that X^{10} does not admit a Spin structure. By the Gromov-Lawson theorem [83] [84], X^{10} admits a metric of positive scalar curvature. By [25], the manifold Y^{11} admits an S^1 -metric of positive scalar curvature.

Positive curvature and the kernel of the Dirac operator. If D is the Dirac operator on a Spin manifold M , ∇ is the covariant derivative on the Spin bundle SM , and ∇^* is the adjoint of ∇ , then

$$D^2 = \nabla^* \nabla + \frac{1}{4} \mathcal{R}, \quad (5.1)$$

where \mathcal{R} is the scalar curvature of M . Since the first term on the right hand side is non-negative then the Dirac operator cannot have any kernel when \mathcal{R} is positive [110]. This classical result has a refinement to KO -theory via the α -invariant. See section 5.2.

Motivation for positive scalar curvature for Y^{11} . Let $D^+(Z^{12}, g_Z)$ be the (chiral) Dirac operator with respect to a Riemannian metric g_Z that coincides with the product metric on $Y^{11} \times I$ in a collar neighborhood of the boundary $\partial Z^{12} = Y^{11}$. This is a Fredholm operator when taking APS boundary conditions. Now consider a continuous family of metrics $g_Z(t)$ on Z^{12} . Then in this case the corresponding family of projections $P(t)$ is *not* continuous for those values of the parameter t where an eigenvalue of $D(Y^{11}, g_Y)$ crosses the origin. If g_Y has positive scalar curvature then from the Lichnerowicz argument [110] using the Weitzenböck formula (5.1) shows that the kernel of $D(Y^{11}, g_Y(t))$ is trivial. Hence $D^+(Z^{12}, g_Z(t))$ is a continuous family of Fredholm operators and therefore $\text{Ind } D^+(Z^{12}, g_Z(t))$ is independent of t .

One motivation for positive scalar curvature for Z^{12} is the following. If g_Z has positive scalar curvature then $\text{Ind } D^+(Z^{12}, g_Z(t))$ vanishes [12].

Effect of the scalar curvature in the Spin^c case. Let $(M, g, \sigma^c, \omega^c)$ be a Spin^c -Dirac manifold with metric g of scalar curvature \mathcal{R} , Spin^c structure σ^c and a connection ω^c on the canonical $U(1)$ bundle $\xi(\sigma^c)$. Let D_ζ be the Dirac operator on $(M, g, \sigma^c, \omega^c)$ twisted with a bundle (ζ, ∇^ζ) . Let Ω^ζ and Ω^α be the principal curvature forms of ∇^ζ and ω^c , respectively. If we write the Ricci curvature as $\Omega^c = \sum_{i \leq (n-1)/2} \lambda_i e_i \wedge e_{[i+(n-1)/2]}$, using the basis $\{e_1, \dots, e_n\}$, then a ‘norm’ is defined as [94] $\|\Omega^c \otimes 1\| = \sum_{i \leq (n-1)/2} |\lambda_i|$. Hence this is, in a sense, a measure of the scalar functions in front of the Ricci curvature written in a Cartan basis.

If $\frac{1}{4}\mathcal{R} - \|\Omega^c \otimes 1 + 1 \otimes \Omega^\zeta\|$ is positive somewhere and nonnegative everywhere on M then [94] [110] $\text{Ind } D_\zeta^+ = 0$; on closed manifolds, $\ker D_\zeta = 0$. Let \mathcal{R}_Y and \mathcal{R}_X be the scalar curvatures of g_Y and g_X on Y^{11} and X^{10} , respectively.

1. *Effect of scalar curvature of Y^{11} .* If $\mathcal{R}_Y(y) > 4\|\Omega_Y \otimes 1 + 1 \otimes \pi^* \Omega^E\|(y)$ for all $y \in Y^{11}$ then the index of the Dirac operator $D_{\pi^*E}^+$ vanishes.

2. *Effect of scalar curvature of X^{10} .* Since $\lim_{t \rightarrow 0} \mathcal{R}_{(Y, g_Y^t)} = \mathcal{R}_X$ and $\mathcal{R}_X \geq \mathcal{R}_{(Y, g_Y^t)}$ then if $\mathcal{R}_X(x) > 4\|\Omega_Y \otimes 1 + 1 \otimes \Omega^E\|(x)$ for all $x \in X^{10}$ then $\lim_{t \rightarrow 0} \text{Ind } D_{\pi^*E}^+(g_Z^t) = 0$. In the case when the Spin^c -Dirac

structure on Y^{11} is bounding then, from [22] [23], (3.33) holds without the mod \mathbb{Z} requirement. Similarly, for (3.41) when the Spin^c -Dirac structure on Y^{11} is strictly equivariant. This is viewed as the classical analog (i.e. at the level of Lagrangians and actions) of the semi-classical statements in section 3.8.1 and section 3.8.2 (which are given there, at least implicitly, at the level of partition functions). This is in harmony with the fact that large scalar curvature takes us further and further into the classical regime.

Ricci curvature and genera. Conditions on Ricci curvature are generically stronger than conditions on the scalar curvature. One example is the Stolz-Höhn Conjecture [155], which is the following statement. Let M be a String manifold, i.e. a manifold admitting a lift of the Spin bundle to a String bundle. If M admits a metric of positive Ricci curvature then the Witten genus $\varphi_W(M)$ vanishes. Now any Spin manifold of positive Ricci curvature has positive scalar curvature and, hence, vanishing \hat{A} -genus. This implies the conjecture in dimension $4k < 24$ or dimension 28. As seen above, the main sources of positive Ricci curvature include Kähler geometry (any Kähler manifold with positive first Chern class carries a metric of positive Ricci curvature), Lie groups, and homogeneous spaces (a compact homogeneous space admits an invariant metric of positive Ricci curvature if its fundamental group π_1 is finite [26]).

Scalar curvature and the ten-dimensional partition function. The partition function in ten dimensions is well-defined when $\langle x \otimes \bar{x}, [X^{10}] \rangle = 0 \in KO_{10}(\text{pt}) \cong \mathbb{Z}_2$ (see section 5.2). In the case when X^{10} is a connected simply connected Spin manifold, we use [144] to deduce that the KO fundamental class $[X^{10}]$ is zero in $KO_{10}(\text{pt}) \cong \mathbb{Z}_2$ if and only if X^{10} admits a metric of positive scalar curvature. Therefore, the partition function is well-defined if and only if X^{10} admits a metric of positive scalar curvature. Hence, in the simply connected case, anomalies arise only for manifolds not admitting metrics of positive scalar curvature. However, a refinement is needed when the fundamental group is nontrivial (see section 6).

5.2 The mod 2 index, the \hat{A} -genus, and the α -invariant

The K-theoretic description of the partition function of the Ramond-Ramond fields in type IIA string theory, reduced from eleven dimensions on a circle, leads to an anomaly given by the mod 2 index of the Dirac operator [54]. We provide a characterization of this within our context.

We first recall some setting from [54]. Consider the product case, $Y^{11} = X^{10} \times S^1$ with the C-field C_3 with field strength G_4 , both pullbacks from X^{10} . In this case, there is an orientation-reversing symmetry under which the term $I_{CS} = \int_{Y^{11}} G_4 \wedge G_4 \wedge C_3$ reverses sign so that the phase, which contains the factor $\exp(iI_{CS})$, is complex-conjugated. This implies that the phase is \mathbb{Z}_2 -valued. In terms of the index theorem, the Dirac operator changes sign under the reflection of one coordinate so that the nonzero eigenvalues appear in pairs $(\lambda, -\lambda)$, implying that the eta invariants are zero. The corresponding analysis for the zero modes shows that the E_8 part of the phase (1.1) reduces to $\Phi_a = (-1)^{f(a)}$, where $f(a)$ is the mod 2 index of the Dirac operator. Here a is the class of the E_8 bundle. The breaking of this bundle on Y^{11} via $E_8 \supset (SU(5) \times SU(5))/\mathbb{Z}_5$ allows for relating to complex bundles on X^{10} , and this is extended to K-theory.

The mod 2 index from twelve dimensions The K-theoretic partition function in [54] depended on a mod 2 index with values in $KO(X^{10})$. The ten-dimensional description of the KO-theoretic class is as the tensor product $V \otimes \bar{V}$, where V is a complex vector bundle and \bar{V} its conjugate, corresponding in the homological setting, respectively, to D-branes and anti-D-branes. The mod 2 index is just $\text{Ind}(D_{V \otimes \bar{V}}) \bmod 2$, which is a topological invariant in ten dimensions [15].

We provide a description of the mod 2 index entering the partition function using twelve-manifolds which are closed, i.e. without the use of the lift to M-theory. Let M^{12} be a *closed* Spin^c twelve-manifold. Our X^{10} will be considered as a submanifold of M^{12} as follows. Consider the projection $\text{Spin}^c(12) \rightarrow SO(2)$ given by the determinant (3.2). At the level of bundles we get the determinant bundle of M^{12} . The ten-dimensional manifold X^{10} is taken to be a (codimension two) submanifold dual to the determinant bundle with the

inclusion being $i : X^{10} \rightarrow M^{12}$. Since the dimension of X^{10} is divisible by 2 mod 8, then we can apply a result of [65]. For every $\psi \in KO(M^{12})$, the KO-characteristic number

$$\langle i^*\psi, [X^{10}] \rangle_{KO} \in \mathbb{Z}_2 \quad (5.2)$$

is the reduction mod 2 of the index in twelve dimensions

$$\left\langle \text{Ph}(\psi) \exp(e/2) \hat{A}(TM^{12}), [M^{12}] \right\rangle \in \mathbb{Z}, \quad (5.3)$$

where Ph is the Pontrjagin character (composition of realification with the Chern character) and $e \in H^2(M^{12}; \mathbb{Z})$ is the Euler class of the determinant bundle. We can then take $E \otimes \bar{E} = i^*\psi$.

5.2.1 The \hat{A} -genus

The \hat{A} -genus is usually defined for manifolds of dimension $4k$. However, there is a lift to KO-theory, which allows an extension to dimensions $8k+1$ and $8k+2$. Thus, for us, this genus will be important in dimensions 1, 2, 9, and 10. Furthermore, the value of the \hat{A} -genus in these dimensions will in general depend on the Spin structure chosen, whenever the manifold admits more than one Spin structure.

Let $D = D(M, \sigma)$ be the Dirac operator defined by a Spin structure σ on a closed manifold M of dimension m . Then, following [79], we consider the following cases.

1. Suppose that $m \equiv 0 \pmod{4}$. Decompose $D = D^+ + D^-$ into the chiral Dirac operators and define $\hat{A}(M, \sigma) = \text{ind}(D^+)$. Inequivalent Spin structures are twisted by flat real line bundles (as in section 2.2). This does not affect the index density and hence $\hat{A}(M, \sigma) = \hat{A}(M)$ is independent of the Spin structure. Furthermore, $\dim \ker(D)$ is even in this case.
2. Suppose that $m \equiv 1 \pmod{8}$. Let $\hat{A}(M, \sigma) \in \mathbb{Z}_2$ be the mod 2 reduction of $\dim \ker(\hat{A}(M, \sigma))$.
3. Suppose $m \equiv 2 \pmod{8}$. The index of the Spin complex is zero. Therefore, $\dim \ker(\hat{A}(M, \sigma)) = 2\dim(D^+(M, s))$ is even. Let $\hat{A}(M, \sigma) \in \mathbb{Z}_2$ be the mod 2 reduction of $\frac{1}{2}\dim \ker(\hat{A}(M, \sigma))$.
4. Set $\hat{A}(M, \sigma) = 0$ otherwise.

The first case is the ‘usual’ \hat{A} -genus, while the second and third are α -invariants (which will be discussed in more detail in section 5.2.2). Hence, the above can also be called *generalized \hat{A} -genera*.

We start with examples of the first case and then recall the examples from section 2.1.

Example 1: Hypersurfaces in projective space. Let $V^n(d)$ denote the nonsingular complex hypersurface of degree d in $\mathbb{C}P^n$. In homogeneous coordinates $[Z_0, \dots, Z_{n+1}]$, $V^n(d)$ is given as the zeros of a homogeneous polynomial $P(Z_0, \dots, Z_{n+1})$ of degree d . The diffeomorphism class of $V^n(d)$ is uniquely determined by the integers n and d . The first Chern class of $V^n(d)$ for $n > 1$ is $c_1 = (n+2-d)x$, where x is the canonical generator of $H^2(V^n(d); \mathbb{Z})$, i.e. the Kähler form induced from $\mathbb{C}P^{n+1}$. The Spin condition means that w_2 is the mod 2 reduction of the integral class c_1 , so that in order for $V^n(d)$ to be Spin, c_1 should be even. This is equivalent to saying that $n+d$ should be even. When $n = 2m$ is even then [107]

$$\hat{A}(V^{2m}(d)) = \frac{2^{-2m}d}{(2m+1)!} \prod_{k=1}^m (d^2 - (2k)^2). \quad (5.4)$$

Thus, each of the Spin manifolds $V^{2m}(2d)$, for $d > n$, has nonzero \hat{A} .

- For $m = 1$, i.e. for $\mathbb{C}P^3$: $\hat{A}(V^2(2d)) = \frac{d}{3}(d^2 - 1)$.
- For $m = 2$, i.e. for $\mathbb{C}P^5$: $\hat{A}(V^4(2d)) = \frac{d}{60}(d^2 - 1)(d^2 - 4)$.
- In general $\hat{A}(V^{2k}(2k+2)) = 2$.

Example 2: Bott manifold. Let b be a generator of $KO_8(\text{pt}) = KO_8(\mathbb{R}) \cong \mathbb{Z}$. One can find a simply-connected manifold B^8 of dimension 8 which represents Bott periodicity in KO_* , with $\alpha(B^8) = b$ (see Section 5.2.2). There are many possible choices for B^8 . One such is the Bott manifold, which is a simply-connected Spin manifold with $\hat{A}(B^8) = 1$ [145]. An example of this is a Joyce manifold with $\text{Spin}(7)$ holonomy. Such examples are important in compactifications of M-theory (see e.g. [1] for a survey).

We now consider examples of the first and second cases. Here, dependence on the Spin structure emerges. This discussion will be naturally continued in section 5.2.2.

Example 3: The circle $M = S^1$. Let θ be the usual periodic parameter on the circle, with ∂_θ providing a global trivialization of the tangent bundle TS^1 and defining a Spin structure σ_1 . The associated Spin bundle is trivial and the Dirac operator is $D = -i\partial_\theta$. Thus $\dim \ker(D) = 1$ and $\hat{A}(S^1, \sigma_1) = 1$. There is another Spin structure σ_2 which is defined by twisting σ_1 with the Möbius bundle. There are no harmonic spinors for σ_2 and $\hat{A}(S^1, \sigma_2) = 0$.

Note the following *terminology*: The \hat{A} -genus is a bordism invariant (as will be indicated in the properties below). The first Spin structure σ_1 does not bound. The second Spin structure σ_2 is induced by regarding the circle as the boundary of the disk. This second Spin structure, corresponding to the nontrivial Spin bundle, is called the “trivial” structure since it bounds.

Example 4: The torus $M = T^2$. The two-dimensional torus $T^2 = S^1 \times S^1$ admits 4 Spin structures σ_i , $i = 1, \dots, 4$. The product Spin structure is σ_0 gives the Spin bundle $T^2 \times \mathbb{C}^2$. This has $\dim \ker(D) = 2$, so that $\hat{A}(T^2, \sigma_0) = 1$. For the other three Spin structures on T^2 we have $\hat{A}(T^2, \sigma_j) = 0$, for $j = 1, 2, 3$.

Some properties of the \hat{A} -genus. The \hat{A} -genus satisfies the following properties.

1. *Multiplicative behavior of the \hat{A} -genus.* Let N be a closed Spin manifold of dimension $n = 4k$ and let M be a closed Spin manifold of dimension m . Then

$$\hat{A}(N \times M) = \hat{A}(N) \cdot \hat{A}(M) . \quad (5.5)$$

2. *Relation to the eta-invariant.* If $\varrho \in RU(\pi_1(M))$ and if M is odd-dimensional then, from Atiyah-Patodi-Singer [11], we get

$$\eta(N \times M)(\varrho) = \hat{A}(N) \cdot \eta(M)(\varrho) . \quad (5.6)$$

For example, for the dimensions of interest to us we have:

$$\begin{aligned} \hat{A}(X^{10}) &= \hat{A}(M^8) \cdot \hat{A}(N^2) \\ \hat{A}(X^{10}) &= \hat{A}(M^4) \cdot \hat{A}(Y^4) \cdot \hat{A}(N^2) \\ \eta(N^4 \times M^7)(\varrho) &= \hat{A}(N^8) \cdot \eta(M^3)(\varrho) \\ \eta(N^4 \times M^7)(\varrho) &= \hat{A}(N^4) \cdot \eta(M^7)(\varrho) . \end{aligned} \quad (5.7)$$

This allows us to extend the evaluation of the eta invariant in eleven dimensions to many cases in which the eleven-dimensional manifold is reducible.

3. *\hat{A} is a bordism invariant.* If M is the boundary of a compact Spin manifold N then $\hat{A}(M, \sigma) = 0$. Furthermore, the \hat{A} genus is independent of the Riemannian metric.
4. *Behavior in fiber bundles.* The \hat{A} -genus vanishes on any smooth fiber bundle of closed oriented manifolds provided that the fiber is a Spin manifold and the structure group is a compact connected Lie group which acts smoothly and non-trivially on the fiber [52].

5.2.2 The α -invariant

We saw in the previous section that the definition of the \hat{A} -genus can be extended beyond manifolds of dimension $4k$. We now consider in more detail how that is done. The \hat{A} -genus, being the index of an elliptic operator, admits a lift to some K -group. Indeed, Milnor found a surjective homomorphism α from the Spin cobordism ring Ω_*^{Spin} onto $KO^{-*}(\text{pt})$ such that $\alpha(M) = \hat{A}(M)$ if $n = 8m$, and which captures additional \mathbb{Z}_2 -information in dimensions $n \equiv 1, 2 \pmod{8}$. The α -invariant associates to any n -dimensional Spin manifold (M, σ) an element $\alpha(M, \sigma) \in KO^{-n}(\text{pt})$, where

$$KO^{-n}(\text{pt}) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is divisible by } 4 \\ \mathbb{Z}_2 & \text{if } n \equiv 1, 2 \pmod{8} \\ 0 & \text{otherwise.} \end{cases} \quad (5.8)$$

The map α defines a surjective ring homomorphism from the Spin cobordism ring Ω_*^{Spin} to $\bigoplus_{k \in \mathbb{N}} KO^{-n}(\text{pt})$.

The index of the Dirac operator. The Dirac operator on X^m has an index in $KO^{-m}(\text{pt})$, given by $f_!(1)$, where $f : X \rightarrow \text{pt}$ is the collapsing map. Depending on m , one has [14]

$$\begin{aligned} KO^{-8m}(\text{pt}) &\cong \mathbb{Z} & f_!(1) &= \hat{A}(X) \\ KO^{-(8m+4)}(\text{pt}) &\cong \mathbb{Z} & f_!(1) &= \frac{1}{2}\hat{A}(X) \\ KO^{-(8m+1)}(\text{pt}) &\cong \mathbb{Z}_2 & f_!(1) &= h_D \pmod{2} \\ KO^{-(8m+2)}(\text{pt}) &\cong \mathbb{Z}_2 & f_!(1) &= h_{D+} \pmod{2} \end{aligned} \quad (5.9)$$

Define $\alpha(X) = f_!(1) \in KO^{-n}(X)$ for a Spin manifold X of dimension n . Then

$$\alpha(X \times Y) = \alpha(X) \cdot \alpha(Y). \quad (5.10)$$

Thus, if $\alpha(M, \sigma) \neq 0$ then the Dirac operator has a nontrivial kernel.

Some properties of α .

1. The Spin cobordism class of a manifold is completely determined by its Stiefel-Whitney and KO -characteristic numbers [5]. A fundamental such KO -invariant is the α -invariant.
2. The homomorphism α is an isomorphism for $n \leq 7$. Thus it is not an isomorphism for $n = 8, 9, 10$.
3. Let the fundamental group of a connected compact Spin manifold M of dimension $m \geq 5$ be a spherical space form group. Then M admits a metric of positive scalar curvature iff $\alpha(M) = 0$ [35].

Example: α -invariant of a Riemann surface. In the case of two dimensions, the α -invariant is

$$\alpha(\Sigma_g) \in KO^{-2}(\text{pt}) = \mathbb{Z}_2 = \{0, 1\}. \quad (5.11)$$

Associate to any Spin structure σ on a Riemann surface Σ_g a quadratic function $q_\sigma : H_1(\Sigma_g; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$. Let V be a finite-dimensional \mathbb{Z}_2 -vector space. For any quadratic map $q_\sigma : V \rightarrow \mathbb{Z}_2$ associated to a nondegenerate symmetric bilinear form one defines the Arf invariant

$$\text{Arf}(q) := \frac{1}{\sqrt{\#V}} \sum_{\alpha \in V} (-1)^{q_\sigma(\alpha)}, \quad (5.12)$$

which has values ± 1 , by virtue of quadraticity of q . For a compact surface of genus g , there are

- $2^{2g-1} + 2^{g-1}$ Spin structures for which the Arf invariant of q_σ is $+1$.
- $2^{2g-1} - 2^{g-1}$ Spin structures for which the Arf invariant of q_σ is -1 .

An alternative interpretation of the Arf invariant is $\text{Arf}(q_\sigma) = (-1)^{\dim \mathcal{H}_s}$, where \mathcal{H}_s is the space of holomorphic sections of the Spin bundle $S(\Sigma_g)$ [7]. The α -invariant $\alpha(\Sigma_g, \sigma)$ can be defined via $\text{Arf}(q_\sigma) = (-1)^{\alpha(\Sigma_g, \sigma)}$ [4]. The index theorem gives that for any compact Riemann surface Σ_g with Riemannian metric g and Spin structure σ

$$\alpha(\Sigma_g, \sigma) = \frac{1}{2} \dim_{\mathbb{C}} \ker D \pmod{2}. \quad (5.13)$$

Case genus $g = 0$: The two-sphere S^2 . $\Sigma_0 = S^2$ with Spin structure σ_0 . Then $q_{\sigma_0} : \{0\} \rightarrow \mathbb{Z}_2$, $q_{\sigma_0}(0) = 0$, and the α -invariant vanishes: $\alpha(S^2, \sigma_0) = 0$.

Case genus $g = 1$: The torus. $\Sigma_1 = T^2 = \mathbb{R}^2/\Gamma$ with a Euclidean metric for a lattice $\Gamma \subset \mathbb{R}^2$, $\Gamma = \pi_1(\Sigma_1) = H_1(\Sigma_1; \mathbb{Z})$. Any group homomorphism $\gamma : \Gamma \rightarrow \mathbb{Z}_2 \subset \ker(\text{Spin}(2) \rightarrow SO(2)) \subset \text{Spin}(2) = S^1$ defines an orthogonal action of Γ on $\mathbb{R}^2 \times \text{Spin}(2)$. A $\text{Spin}(2)$ principal bundle is defined by factoring the Γ -action

$$P_{\text{Spin}}(\Sigma_1, g)_\Gamma := \mathbb{R}^2 \times_\gamma \text{Spin}(2). \quad (5.14)$$

A Spin structure σ_γ on Σ_1 is defined via the principal bundle (5.14) and the natural map

$$\begin{aligned} \sigma_\gamma : P_{\text{Spin}}(\Sigma_1, g) = \mathbb{R}^2 \times_\gamma \text{Spin}(2) &\longrightarrow P_{SO}(\Sigma_1, g) = (\mathbb{R}^2/\Gamma) \times SO(2) \\ [(x, z)]_\gamma &\longmapsto (x + \Gamma, z^2). \end{aligned} \quad (5.15)$$

Two Spin structures σ_{γ_1} and σ_{γ_2} are equivalent if and only if $\gamma_1 = \gamma_2$. Let \bar{v} denote the image of $v \in \Gamma$ in $H_1(\Sigma_1; \mathbb{Z}_2) = \Gamma \otimes_{\mathbb{Z}} \mathbb{Z}_2$. The quadratic form q of the Spin structure associated to γ satisfies

$$q(\bar{v}) \cong \begin{cases} \gamma(v) + 1 & \text{for } \bar{v} \neq 0 \\ \gamma(v) = 0 & \text{for } \bar{v} = 0, \end{cases} \quad (5.16)$$

so that

$$\alpha(\Sigma_1, \sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is the Spin structure associated to the trivial map } \gamma, \\ 0 & \text{otherwise.} \end{cases} \quad (5.17)$$

One can be more explicit in this, using elliptic curves [104]. There are 4 Spin structures on the torus. Let $\mathbb{C}/\{2\omega_1, 2\omega_3\} = \text{Jac}(T^2)$ be the Jacobian for T^2 , and let $e_i = \wp(\omega_i)$, ($i = 1, 2, 3$), where $\omega_2 = \omega_1 + \omega_3$. Then $h(u) = (\wp(u), \wp'(u))$ is a conformal diffeomorphism from the Jacobian to the Riemann surface Σ_1 defined by

$$\omega^2 = 4(z - e_1)(z - e_2)(z - e_3). \quad (5.18)$$

Then the four distinct Spin structures are defined by the four differentials

$$\begin{aligned} du &= dz/\omega \\ (\wp(u) - e_i)du &= (z - e_i)dz/\omega. \end{aligned} \quad (5.19)$$

With a_i the generator of $H_1(T^2; \mathbb{Z}_2)$ defined by $a_i : [0, 1] \rightarrow \text{Jac}(T^2)$, $a_i(t) = 2t\omega_i$, then the values of the Arf invariant for the last three of the four Spin structures in (5.19) are 1, whereas, for du :

$$q_\sigma(0) = 0, \quad q_\sigma(a_i) = +1 \quad \text{for } i = 1, 2, 3, \quad (5.20)$$

so that $\text{Arf}(q_\sigma) = -1$. This gives an α -invariant which is odd in this case, $\alpha(T^2, \sigma_4) = 1$.

5.2.3 Ten-dimensional manifolds: products and bundles

Ten-dimensional manifolds which are products. Manifolds with free S^1 -action and nonvanishing α -invariant can be obtained by taking the product of S^1 , having the nontrivial Spin structure, with any 8-dimensional Spin manifold with an odd \hat{A} -genus [120]. (Simply connected examples are given in [108]).

Thus we have the following examples of (reducible) ten-dimensional manifolds for which the α -invariant is possibly nonzero:

$$\begin{aligned}
\alpha(X^{10}) &= h_{D+} \bmod 2 := r_2(h_{D+}) \\
\alpha(X^9 \times S^1) &= \alpha(X^9)\alpha(S^1) = (h_D(X^9) \bmod 2)(h_D(S^1) \bmod 2) \\
&= r_2(h_D(X^9))r_2(h_D(S^1)) \\
\alpha(X^8 \times \Sigma_g) &= \hat{A}(X^8)(h_{D+}(\Sigma) \bmod 2) = \hat{A}(X^8)r_2(h_{D+}(\Sigma)) \\
\alpha(X^4 \times Y^4 \times \Sigma_g) &= \frac{1}{4}\hat{A}(X^4)\hat{A}(Y^4)r_2(h_{D+}(\Sigma)) \\
\alpha(X^4 \times \Sigma_{g_1} \times \Sigma_{g_2} \times \Sigma_{g_3}) &= \frac{1}{2}\hat{A}(X^4)r_2(h_{D+}(\Sigma_{g_1}))r_2(h_{D+}(\Sigma_{g_2}))r_2(h_{D+}(\Sigma_{g_3})) \\
\alpha(\Sigma_{g_1} \times \Sigma_{g_2} \times \Sigma_{g_3} \times \Sigma_{g_4} \times \Sigma_{g_5}) &= r_2(h_{D+}(\Sigma_{g_1}))r_2(h_{D+}(\Sigma_{g_2}))r_2(h_{D+}(\Sigma_{g_3}))r_2(h_{D+}(\Sigma_{g_4}))r_2(h_{D+}(\Sigma_{g_5})) .
\end{aligned}$$

In addition, we can have factors with products of circles

$$\alpha \left(M^{4n} \times \prod_{i=1}^{10-4n} (S^1)^i \right) = \epsilon' \hat{A}(M^{4n}) \prod_{i=1}^{10-4n} (h_D(S_i^1) \bmod 2)^i , \quad (5.21)$$

where ϵ' equals $\frac{1}{2}$ for $n = 4$ and equals 1 for $n = 8$, and where S_i^1 denotes the i th circle.

For example, for $X^{10} = M^8 \times (S^1)^2$, where S^1 is the circle with the nonbounding Spin structure, $\alpha(X^{10}) = 0$ is equivalent to the condition $\hat{A}(M^8) \equiv 0 \bmod 2$. Examples of M^8 are the quaternionic projective plane $\mathbb{H}P^2$, for which $\hat{A}(\mathbb{H}P^2)_8 = 0$. Thus $\alpha(\mathbb{H}P^2 \times S^1 \times S^1) = 0$. Similarly for the Milnor manifold M_0^8 [120], which is a Spin manifold with $\hat{A}(M_0^8) = 1$.

Ten-dimensional manifolds which are bundles. Let $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$ be a short exact sequence of real vector bundles. let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. If V_i and V_j are Spin then V_k has a natural Spin structure. Thus consider the following situation. Let $F^n \rightarrow X^{10} \xrightarrow{p} M^{10-n}$ be a differentiable fiber bundle with Spin base and total space. The Spin structures on the base and the total space induce a Spin structure on the tangent bundle along the fibers TF^n . For $f : M^{10-n} \rightarrow \text{pt}$, the α -invariant of the ten-dimensional manifold is defined as

$$\alpha(X^{10}) = (f_p)_!(1) = f_!(p_!(1)) \in KO^{-10}(\text{pt}) \cong \mathbb{Z}_2 , \quad (5.22)$$

where the direct image homomorphisms are taken relative to the Spin structures on the base and the fiber. This means that if $p_!(1) = 0$, then $\alpha(X^{10}) = 0$. Thus, we have examples with nonzero α (cf. [94])

$$\begin{aligned}
F^9 &\rightarrow X^{10} \rightarrow S^1 \\
F^8 &\rightarrow X^{10} \rightarrow \Sigma_g .
\end{aligned} \quad (5.23)$$

The fiber in the each of the above bundles admits harmonic spinors with respect to some metric.

Cobordism generators for ten-dimensional manifolds which are bundles with $\alpha = 0$. Recall from section 3.5 that one of the two generators of Ω_8^{Spin} is the quaternionic projective plane $\mathbb{H}P^2$. The space $\ker \alpha$ can be represented by total spaces of bundles with fibers given as $\mathbb{H}P^2$ [154]. Note that

$$\begin{aligned}
\ker \alpha &= 0 & \text{in dim } \leq 7 \\
\ker \alpha &= \mathbb{Z} & \text{in dim } 8 ,
\end{aligned} \quad (5.24)$$

where the generator is represented by $\mathbb{H}P^2$. Any closed simply connected spin manifold M with $\alpha(M) = 0$ is Spin bordant to the total space of a bundle with $\mathbb{H}P^2$ as fiber and structural group $\text{Isom}(\mathbb{H}P^2) = PSp(3) =$

$Sp(3)/\mathbb{Z}_2$, the group of isometries of $\mathbb{H}P^2$ [154]. Given such a fiber bundle, one can use the O'Neill formulas to produce a metric of positive scalar curvature on the total space from the standard positive scalar curvature metric on $\mathbb{H}P^2$ and an arbitrary Riemannian metric on the base. Note that the base of such a fiber bundle is compact, so one can squeeze the fibers so that at each point the scalar curvature contribution coming from the fiber dominates the contribution coming from the base. Thus any $\mathbb{H}P^2$ -bundle can be equipped with a metric of positive scalar curvature on its total space. Let $G = PSp(3)$ and $H = P(Sp(2) \times Sp(1))$. Given a manifold X and a map $f : X \rightarrow BG$, let $\hat{X} \rightarrow X$ be the pullback of the fiber bundle

$$\mathbb{H}P^2 \rightarrow G/H \rightarrow BH \xrightarrow{\pi} BG \quad (5.25)$$

via f . Consider the homomorphism $\Psi : \Omega_2^{\text{Spin}} \rightarrow \Omega_{10}^{\text{Spin}}$ by mapping the bordism class of f to the bordism class of \hat{X} . Then [154]

$$\ker \alpha = \text{Im } \Psi, \quad (5.26)$$

that is, if X is a closed Spin manifold with $\alpha(X) = 0$ then X is Spin cobordant to the total space of a fiber bundle with fiber $\mathbb{H}P^2$ and structure group the projective symplectic group $PSp(3)$. Thus, the kernel of α forms a subgroup T_{10} consisting of bordism classes represented by total spaces of $\mathbb{H}P^2$ bundles.

Any Spin manifold of dimension ≥ 8 with $\hat{A}(M) = 0$ is rationally cobordant to an $\mathbb{H}P^2$ -bundle with nontrivial S^3 -action along the fibers. Kreck-Stolz show the much deeper result that any Spin manifold M of dimension ≥ 8 with $\alpha(M) = 0$ is *integrally* cobordant to the total space of an $\mathbb{H}P^2$ -bundle with structure group $PSp(3)$ [102]. There is a canonical isomorphism

$$KO_n(X) \cong \left(\bigoplus_k \Omega_{n+8k}^{\text{Spin}}(X) \right) / \sim, \quad (5.27)$$

where the equivalence is generated by

1. $[E, f \circ p] \sim 0$ if $p : E \rightarrow M$ is an $\mathbb{H}P^2$ bundle over a Spin manifold M and f is a map from M to X .
2. $[M, f] \sim [M \times B, f \circ \text{pr}_1]$, where B is the Bott manifold and pr_1 denotes the projection to the first factor.

Given a space M , let $T_*(M)$ be the subgroup of $\Omega_*^{\text{Spin}}(M)$ represented by pairs $(X^{10}, f \circ p)$ where $X^{10} \rightarrow \Sigma_g$ is an $\mathbb{H}P^2$ bundle and $f : \Sigma_g \rightarrow M$ is a map. Let $b \in \Omega_8^{\text{Spin}}(\text{pt})/T_8(\text{pt}) \cong KO_8(\text{pt}) \cong \mathbb{Z}$ be the generator, i.e., the class represented by the Bott manifold. Then the homomorphism $\rho \circ D_* : \Omega_*^{\text{Spin}}(M) \rightarrow KO_*(M)$ induces an isomorphism

$$\Omega_*^{\text{Spin}}(M)/T_*(M)[b^{-1}] \cong KO_*(M). \quad (5.28)$$

This makes a connection between 2 and 10 dimensions. In particular, we can take a type IIA string wrapping a Riemann surface, which we take as the base of the above fibration.

Eleven-dimensional manifolds which are total spaces of $\mathbb{H}P^2$ bundles. Consider a Spin eleven-dimensional manifold Y^{11} which is an $\mathbb{H}P^2$ bundle π over a Spin three-manifold M^3 . Let there be a map from Y^{11} to some other space (e.g. classifying space) \mathcal{M} which factors through M^3 , so that we have the following diagram

$$\begin{array}{ccc} \mathbb{H}P^2 & \longrightarrow & Y^{11} \\ & & \downarrow \pi \\ & & M^3 \end{array} \quad \begin{array}{c} \searrow \\ \xrightarrow{f} \end{array} \quad \mathcal{M} \quad (5.29)$$

Let $T_{11}(\mathcal{M})$ be the subgroup of $\Omega_{11}^{\text{Spin}}(\mathcal{M})$ consisting of bordism classes $[Y^{11}, f \circ \pi]$. Let $\tilde{T}_{11}(\mathcal{M})$ be the subgroup with the additional assumption that $[M^3, f] = 0 \in \Omega_3^{\text{Spin}}(\mathcal{M})$.

- **Case $\mathcal{M} = \text{pt}$:** Here we have $\Omega_{11}^{\text{Spin}}(\text{pt})$, which is zero. So there are no obstructions in this case.
- **Case $\mathcal{M} = BE_8$:** We have an E_8 bundle on Y^{11} which we are asking to extend to twelve dimensions. However, since by a result of Stong, $\Omega_{11}^{\text{Spin}}(K(\mathbb{Z}, 4)) = 0$, the above subgroups will be trivial and so there is not much to do in this case.
- **Case $\mathcal{M} = B\pi_1(Y^{11})$:** Since the fiber $\mathbb{H}P^2$ is simply connected, then $\pi_1(Y^{11})$ and $\pi_1(M^3)$ are isomorphic. This implies that there is indeed a map $f : M^3 \rightarrow B\pi_1(Y^{11}) = B\pi_1(M^3)$, which is the classifying map for the fundamental group.

As in the ten-dimensional case, this provides a link between dimension eleven (M-theory) and three (which we can take to be where the membrane wraps or lives).

Ten-dimensional manifolds which are total spaces of $\mathbb{H}P^2$ bundles. Consider a Spin ten-dimensional manifold M^{10} which is an $\mathbb{H}P^2$ bundle π over an oriented Riemann surface Σ_g . We take a map from M^{10} to \mathcal{M} which factors through Σ_g , so that we have the following diagram

$$\begin{array}{ccc} \mathbb{H}P^2 & \longrightarrow & M^{10} \\ & & \downarrow \pi \\ & & \Sigma_g \end{array} \quad \begin{array}{c} \searrow \\ \xrightarrow{f} \end{array} \quad \begin{array}{c} \\ \mathcal{M} \end{array} \quad (5.30)$$

Let $T_{10}(\mathcal{M})$ and $\tilde{T}_{10}(\mathcal{M})$ be the subgroups as in the eleven-dimensional case. We consider ten-dimensional Spin cobordism in the setting where M-theory is taken on an eleven-dimensional Spin manifold Y^{11} with boundary $M^{10} = \partial Y^{11}$ as in [53].

- **Case $\mathcal{M} = \text{pt}$:** Here we have $\Omega_{10}^{\text{Spin}}(\text{pt}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The α -invariant has this as a domain. Here $T_{10}(\text{pt}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- **Case $\mathcal{M} = BE_8$:** We have an E_8 bundle on X^{10} which we wish to extend to eleven dimensions. Unlike the case $n = 11$, here $\Omega_{10}^{\text{Spin}}(K(\mathbb{Z}, 4)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- **Case $\mathcal{M} = B\pi_1(M^{10})$:** Since the fiber $\mathbb{H}P^2$ is simply connected, then $\pi_1(M^{10})$ and $\pi_1(\Sigma_g)$ are isomorphic. This implies that there is indeed a map $f : \Sigma_g \rightarrow B\pi_1(X^{10}) = B\pi_1(\Sigma_g)$, which is the classifying map for the fundamental group.

Homotopy spheres. Let Σ^n be an n -dimensional homotopy sphere, i.e. a compact differentiable manifold which is homotopy equivalent to the n -sphere S^n . Then Σ^n is cobordant to zero but not necessarily Spin cobordant to zero. In fact, the homotopy n -spheres form an abelian group Θ_n under the operation of connected sum, and

$$\alpha : \Theta_n \rightarrow KO^{-n}(\text{pt}) \quad (5.31)$$

is a homomorphism. For $n \equiv 1$ or $2 \pmod{8}$ and $n > 8$, the homomorphism $\alpha : \Theta_n \rightarrow \mathbb{Z}_2$ is surjective (see [107]). This means that we can find exotic spheres in dimensions nine and ten for which the alpha invariant is 1, the nontrivial element in \mathbb{Z}_2 . This can serve as a source of an anomaly for the partition function.

Exotic spheres. Among the manifolds for which α is nonzero is half the exotic spheres in dimensions 9 and 10. In these dimensions every compact Spin manifold is homeomorphic to a manifold which does not carry a positive scalar curvature. This can be seen as follows (see [107]). Let X be a Spin k -manifold with $k = 9$ or 10 , and let Σ be an exotic k -sphere with $\alpha(\Sigma) \neq 0$. Since $\alpha(X \# \Sigma) = \alpha(X) + \alpha(\Sigma)$, then X and $X \# \Sigma$ are compact Spin manifolds with positive scalar curvature, so that their α -invariant vanishes. Thus every Spin manifold of dimension 9 or 10 is homeomorphic to one with nontrivial α -invariant. This means that there is an abundance of manifolds which could lead to an anomaly.

Which exotic spheres have a non-trivial α -invariant? The α -invariant constitutes a surjective group homomorphism from the group Θ_n of homotopy n -spheres (with the addition induced by the connected sum operation) onto the group \mathbb{Z}_2 , provided $n \equiv 1, 2 \pmod{8}$ and $n \geq 9$. Roughly speaking, Adams has shown [2] that a non-trivial α -invariant in dimensions $n \equiv 1, 2 \pmod{8}$ can always be realized through a stably framed closed manifold, while Milnor has shown [121] that one can alter an accordingly framed manifold to a homotopy sphere by a sequence of surgeries without changing its α -invariant, provided $n \geq 9$ (and $n \equiv 1, 2 \pmod{8}$).

Interpretation of α in terms of connective real K-theory. The α -invariant $\alpha(X)$ can be interpreted as the image of the class $[X]$ under a natural transformation of generalized cohomology theories [146]. Let $KO_*(X)$ and $ko_*(X)$ denote the periodic and connective real K-theory of a space X , respectively. Then there are natural transformations

$$\begin{aligned} \Omega_*^{\text{Spin}} &\xrightarrow{D} ko_*(X) \xrightarrow{\text{per}} KO_*(X) \\ [X, f] &\longmapsto f_*([X]_{ko}) \longmapsto \text{per} \circ D(X) = \alpha(X), \end{aligned} \quad (5.32)$$

where $[X]_{ko} \in ko_*(X)$ denotes the ko -fundamental class of X determined by the Spin structure. The natural transformation per induces an isomorphism $ko_*(\text{pt}) \cong KO_n(\text{pt})$ for $n \geq 0$.

5.3 Types of metrics and boundary conditions

We have considered M-theory on an eleven-dimensional manifold Y^{11} . This is a total space of a circle bundle over X^{10} , in relating to type IIA string theory in ten dimensions. On the other hand, this is the boundary of a twelve-manifold Z^{12} , in studying the topological aspects of the partition function. Of course, we could use both at the same time and consider Z^{12} as the total space of a two-disk bundle over X^{10} . The task then is to study effects of the geometry, and in particular, the effects of choice of metrics on each one of the above spaces. Furthermore, one could ask whether there are preferred metrics which are favored by both the fibration structure on Y^{11} as well as by (a variation on) the Atiyah-Patodi-Singer index theorem. We will make use of the results in [87] on metrics and those in [109] on the corresponding index theorems.

Let Z^{12} be a smooth twelve-dimensional Spin manifold with boundary $\partial Z^{12} = Y^{11}$. Let z be a boundary function such that on the boundary we have $z = 0$ and $dz \neq 0$. Let g_Y be a smooth metric on Y^{11} . Assume that both Z^{12} and X^{10} are Spin, and fix a Spin structure on each. There is an induced structure on the fibers $\pi^*(X^{10}) := S_x^1 \subset Y^{11}$. The Spin bundles on Z^{12} , Y^{11} , and S_x^1 are denoted $SZ = S^+(Z) \oplus S^-(Z)$, SY and SS_x^1 , respectively. Let $E \rightarrow Z^{12}$ be a Hermitian complex vector bundle corresponding to an E_8 principal bundle endowed with a unitary connection. Fixing a metric g_Z on Z^{12} , we have a twisted Dirac operator

$$D_{g_Z}^+ : C^\infty(Z^{12}, E \otimes S^+(Z)) \rightarrow C^\infty(Z^{12}, E \otimes S^-(Z)). \quad (5.33)$$

The boundary operator D^Y induces a family of operators $\{D_x^Y\}_{x \in X}$, where each D_x^Y acts on the space of sections $C^\infty(\pi^{-1}(x), E \otimes S_x^Y)$. See section 7 for more on families.

Case Z^{12} arbitrary. Here we have the following two types of metrics:

1. *Exact b-metric* g_Z^b on the interior of Z^{12} : This takes the form $g_Z^b = \frac{1}{z^2} dz^2 + g_Y$ in the neighborhood of the boundary. Setting $z = e^{-t}$ gives the metric $dt^2 + g_Y$ on $\mathbb{R}^+ \times \partial Z^{12}$, so that it has cylindrical ends. The index theorem corresponding to the Dirac operator $D_{g_Z^b}$, defined using the metric g_Z^b , is the Atiyah-Patodi-Singer index theorem.
2. *Exact cusp c-metric or scattering metric* g_Z^c on Z^{12} : $g_Z^c = \frac{1}{z^4} dz^2 + \frac{1}{z^2} g_Y$. Taking $z = \frac{1}{r}$ gives $dr^2 + r^2 g_Y$ with $r \rightarrow \infty$, which is the standard metric of the infinite end of a cone and corresponds to the asymptotically locally Euclidean (ALE) class of gravitational instantons, such as the Eguchi-Hanson metric. This is thus appropriate for the Kaluza-Klein monopole in M-theory. The corresponding index theorem in this case, the index theorem corresponding to the Dirac operator $D_{g_Z^c}$, defined using the c -metric, reduces to the Atiyah-Patodi-Singer index theorem in a simple way.

Case $\partial Z^{12} = Y^{11}$ is a fibration. Consider the case when the eleven-dimensional boundary is the total space of a fibration $\pi : Y^{11} \rightarrow B^n$, with $(11-n)$ -dimensional fiber F . In this case we have the index theorem relating Z^{12} to X^{10} , i.e. respecting the fibration structure of Y^{11} . The tangent bundle decomposes as

$$\begin{aligned} T(\partial Z^{12}) &= T_V(\partial Z^{12}) \oplus T_H(\partial Z^{12}) \\ &= T(\partial Z^{12}/X^{10}) \oplus \pi^*(TX^{10}). \end{aligned} \quad (5.34)$$

Let k_Y be a symmetric two-tensor on ∂Z^{12} which restricts to a metric on each fiber F . In this case, there are two natural types of metrics:

1. *Fibered boundary metric g_Z^Φ on Z^{12} :* $g_Z^\Phi = \frac{1}{z^4}dz^2 + \frac{1}{z^2}\pi^*g_X + k_Y$. Letting $z = 1/r$ gives the metric $dr^2 + r^2\pi^*g_X + k_Y$, which is the ALF (asymptotically locally flat) and the ALG classes⁶ of gravitational instantons such as Taub-NUT space. the index of the Dirac operator using this metric, from [109], is

$$\text{Ind}(D_\Phi^+) = \int_{Z^{12}} \hat{A}(Z^{12}, g_Z^\Phi) \wedge \text{ch}E - \frac{1}{2} \int_{X^{10}} \hat{A}(X^{10}, g_X) \wedge \hat{\eta}, \quad (5.35)$$

where $\hat{\eta} \in \Omega^*(X^{10})$ is the Bismut-Cheeger eta form for the boundary family $(D_x^\partial)_{x \in X^{10}}$. Eta forms will be discussed in detail in section 7.

2. *Cusp fibered d -metric g_Z^d on Z^{12} :* $g_Z^d = \frac{1}{z^2}dz^2 + \pi^*g_X + z^2k_Y$. Letting $z = e^{-t}$ gives the metric $dt^2 + \pi^*g_X + e^{-2t}k_Y$, which is the standard form for ∂Z^{12} a torus bundle over a torus. The d -metric is related to the Φ -metric by conformal transformations $g_Z^d = z^2g_Z^\Phi$, so that $\hat{A}(Z^{12}, g_Z^\Phi) = \hat{A}(Z^{12}, g_Z^d)$ pointwise. Then, the results of [126] [160] [109] imply that the index for the d -metric coincides with that of the Φ -metric

$$\text{Ind}(D_d^+) = \int_{Z^{12}} \hat{A}(Z^{12}, g_Z^d) \wedge \text{ch}E - \frac{1}{2} \int_{X^{10}} \hat{A}(X^{10}, g_X) \wedge \hat{\eta}. \quad (5.36)$$

The above results can be generalized to the case when Y^{11} is a general fiber bundle $F^{11-n} \rightarrow Y^{11} \rightarrow B^n$.

Dependence of space of harmonic spinors on the metric in 10 and 11 dimensions. From [94], one can get examples of Spin 10-manifolds for which the space of harmonic spinors depends on the metric: $S^7 \times S^3$. Of course we can also take quotients by finite groups $S^7/\Gamma_1 \times S^3/\Gamma_2$ and/or consider nontrivial bundles. In eleven dimensions one has $S^8 \times S^3$ and similarly their quotients and nontrivial bundles.

6 Non-simply Connected Manifolds

6.1 Non-simply connected ten-dimensional manifolds $\pi_1(X^{10}) \neq 1$

If X^{10} is a Spin manifold with nontrivial fundamental group then we can consider a refinement to a cobordism class which takes $\Gamma = \pi_1(X^{10})$ into account. Consider maps $X^{10} \rightarrow B\Gamma \times B\text{Spin}$.

C^* -algebras. Let $\Omega_n^{\text{Spin}}(B\Gamma)$ be the bordism group of closed Spin manifolds M of dimension n with a reference map to $B\Gamma$. Let $C_r^*(\Gamma; \mathbb{R})$ be the real reduced group C^* -algebra of Γ , and let $KO_n(C_r^*(\Gamma; \mathbb{R}))$ be its topological K-theory. Given an element $[u : M \rightarrow B\Gamma] \in \Omega_n^{\text{Spin}}(B\Gamma)$, we can take the $C_r^*(\Gamma; \mathbb{R})$ -valued index of the equivariant Dirac operator associated to the Γ -covering $\tilde{M} \rightarrow M$. Recall that a Bott manifold is any simply connected closed Spin manifold B^8 of dimension 8 with \hat{A} -genus $\hat{A}(B^8) = 1$. This manifold is not unique. B^8 geometrically represents Bott periodicity in KO-theory [146]. A generator is $\text{Ind}_{C_r^*(\{1\}; \mathbb{R})}(B^8) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$ and the product with this element induces the Bott periodicity isomorphism

$$KO_n(C_r^*(\Gamma; \mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(\Gamma; \mathbb{R})). \quad (6.1)$$

⁶The terminology ALG in the literature is meant to ‘mimic’ ALE, that is alphabetically G is the letter right after F.

In particular

$$\mathrm{Ind}_{C_r^*(\Gamma; \mathbb{R})}(M) = \mathrm{Ind}_{C_r^*(\Gamma; \mathbb{R})}(M \times B^8) \quad (6.2)$$

if we use the identification (6.1) via Bott periodicity (see e.g. [101]). Relevant examples for us are, as in section 5.2.3, when M is a Riemann surface Σ_g or a Spin 3-manifold M^3 for type IIA and M-theory, respectively. This again is a way to relate phenomena in 2 or 3 dimensions to ones in 10 or 11 dimensions, via a sort of dimensional reduction/lifting governed by Bott periodicity when the internal space is, for instance, an eight-manifold of Spin(7) holonomy.

Flat vector bundles. Consider the Mishchenko-Fomenko bundle [123]

$$\mathcal{V} := \tilde{X} \times_{\Gamma} C^*\Gamma, \quad (6.3)$$

which is a bundle of right $C^*\Gamma$ -modules. This is a flat bundle, so that it has no effect on characteristic classes. Consider the Spin bundle SX coupled to \mathcal{V} via $SX \otimes \mathcal{V}$, and form the space of sections $\Gamma(X; SX \otimes \mathcal{V})$, on which the Dirac operator acts on the left and $C^*\Gamma$ acts on the right, so that the two actions commute. This gives $\ker D$ as a $C^*\Gamma$ -module, and $[\ker D]$ as a finitely generated projective module, and hence represents a class $[\ker D] \in KO(C^*\Gamma)$.

In general, the mod 2 index of the $\mathcal{C}\ell$ -linear Dirac operator twisted by the Mishchenko-Fomenko bundle \mathcal{V} is an element of $KO^{-n}(C^*\Gamma)$. This is the desired mod 2 index when the fundamental group of X is nontrivial. The above reduces to the known formula when the fundamental group is trivial. This is because of the equalities in that case

$$KO(C^*\Gamma) = KO_{\mathrm{cpct}}(\mathbb{R}) = KO(\mathrm{pt}). \quad (6.4)$$

Here α is the composition

$$\Omega_n^{\mathrm{Spin}}(B\Gamma) \xrightarrow{D} ko_n(B\Gamma) \xrightarrow{p} KO_n(B\Gamma) \xrightarrow{A} KO_n(C_r^*\Gamma), \quad (6.5)$$

where A is the *assembly map*, whose target is the KO-theory of the (reduced) real group X^* -algebra $C_r^*\Gamma$, which is the norm completion of the real group ring $R\Gamma$ (equal to it if Γ is finite).

The Rosenberg-Stolz construction. Let ξ be an auxiliary real vector bundle over the classifying space $B\Gamma$. If $\Gamma = \mathbb{Z}_n$, any complex line bundle over $B\mathbb{Z}_n$ is given by a representation of \mathbb{Z}_n . Also, any element of $H^2(B\mathbb{Z}_n; \mathbb{Z}_2)$ lifts to $H^2(B\mathbb{Z}_n; \mathbb{Z})$. To define equivariant Spin bordism we take ξ to be the trivial line bundle. Taking nontrivial ξ allows one to define twisted Spin bordism groups. Consider triples (M, f, σ) , where f is a Γ structure on M and σ is a Spin structure on $TM \oplus f^*\xi$. Introduce the *bordism relation*: $[(M, f, \sigma)] = 0$ if there exists a compact manifold N with boundary M so that the structures f and σ extend over N . This induces a suitable equivalence class and the *twisted bordism group* $\Omega_m^{\mathrm{Spin}}(B\Gamma, \xi)$ consists of bordism classes of these triples. Disjoint union defines the group structure. Cartesian product makes $\Omega_m^{\mathrm{Spin}}(B\Gamma, \xi)$ into an Ω_*^{Spin} -module.

A Spin structure σ on $TM \oplus f^*\xi$ and a Spin^c structure σ_{ξ}^c on ξ define a natural Spin^c structure σ_M^c on M . If σ_{ξ}^c is a Spin structure σ_{ξ} then we get a Spin structure on M . So there is a homomorphism [145]

$$\Omega_m^{\mathrm{Spin}}(B\Gamma, \xi) \rightarrow \Omega_m^{\mathrm{Spin}^c}(B\Gamma) \quad \text{if } \sigma_{\xi}^c \text{ is a } \mathrm{Spin}^c \text{ structure.} \quad (6.6)$$

If ξ is trivial, then $\Omega_m^{\mathrm{Spin}}(B\Gamma, \xi)$ can be identified with the ordinary Spin bordism group $\Omega_m^{\mathrm{Spin}}(B\Gamma)$. The map that sends (M, f, σ) to (M, σ) which forgets the Γ structure induces a natural forgetful map

$$\Omega_m^{\mathrm{Spin}}(B\Gamma) \rightarrow \Omega_m^{\mathrm{Spin}}. \quad (6.7)$$

Conversely, to any Spin manifold M can be associated a trivial Γ structure. Therefore, there is a direct sum decomposition $\Omega_m^{\mathrm{Spin}}(B\Gamma) = \Omega_m^{\mathrm{Spin}} \oplus \tilde{\Omega}_m^{\mathrm{Spin}}(B\Gamma)$, where the *reduced bordism groups* $\tilde{\Omega}_m^{\mathrm{Spin}}(B\Gamma)$ carry the

essential new information. Note that the even-dimensional reduced bordism groups for $\Gamma = \mathbb{Z}_n$ vanish [79] $\tilde{\Omega}_{2k}^{\text{Spin}}(B\mathbb{Z}_n) = 0$.

Let $\ell = 2^q \geq 4$, let f be a \mathbb{Z}_ℓ structure on X^{10} , and let σ be a Spin structure on $TX^{10} \oplus f^*\xi$. Let σ_X be a Spin structure on X^{10} , and let $\hat{\sigma}$ be the Spin structure σ_X twisted by the representation $\varrho_{\ell/2}$. Since $10 \equiv 2 \pmod{8}$, then [79]

$$\alpha(M, f, \sigma) := \hat{A}(M, \sigma) \oplus \hat{A}(M, \hat{\sigma}) \subset \mathbb{Z}_2 \oplus \mathbb{Z}_2 . \quad (6.8)$$

Example: the circle and the two-torus. Consider the simpler case of dimension two. Let S^1 be the circle with the trivial \mathbb{Z}_ℓ structure and nonbounding Spin structure. Let \bar{S}^1 be the circle with nontrivial \mathbb{Z}_ℓ structure and nonbounding Spin structure. The α -invariants are

$$\alpha(S^1) = 1 \oplus 1 , \quad \alpha(\bar{S}^1) = 1 \oplus 0 . \quad (6.9)$$

Next consider the torus. Let $T^2 = S^1 \times S^1$ and $\bar{T}^2 = \bar{S}^1 \times \bar{S}^1$. The α -invariants are

$$\alpha(T^2) = 1 \oplus 1 , \quad \alpha(\bar{T}^2) = 1 \oplus 0 . \quad (6.10)$$

Further examples leading to ten-dimensional manifolds can be constructed using the above spaces as parts (i.e. fibers or base spaces) of ten-dimensional bundles. The α -invariant on the bordism groups is given by [79]

$$\begin{aligned} \alpha\left(\Omega_{8k+1}^{\text{Spin}}(B\mathbb{Z}_\ell)\right) &= \mathbb{Z}_2 \oplus \mathbb{Z}_2 , & \alpha\left(\Omega_{8k+2}^{\text{Spin}}(B\mathbb{Z}_\ell)\right) &= \mathbb{Z}_2 \oplus \mathbb{Z}_2 , \\ \alpha\left(\tilde{\Omega}_{8k+1}^{\text{Spin}}(B\mathbb{Z}_\ell)\right) &= \mathbb{Z}_2 , & \alpha\left(\tilde{\Omega}_{8k+2}^{\text{Spin}}(B\mathbb{Z}_\ell)\right) &= \mathbb{Z}_2 . \end{aligned} \quad (6.11)$$

Equivariant Spin and Spin^c cobordism. Let X_1^{10} and X_2^{10} be smooth compact 10-dimensional manifolds without boundary and with (Spin, Γ) structure, and similarly for (Spin^c, Γ) structure. X_1^{10} is (Spin, Γ) bordant to X_2^{10} if there exists a smooth compact eleven-dimensional manifold N^{11} so that $\partial N^{11} = X_1^{10} - X_2^{10}$ such that the (Spin, Γ) structure extends over N^{11} . Let $\Omega_{10}^{\text{Spin}}(B\Gamma)$ be the set of (Spin, Γ) bordism equivalence classes.

- Disjoint union gives $\Omega_{10}^{\text{Spin}}(B\Gamma)$ the structure of an abelian group.
- $\Omega_*^{\text{Spin}}(B\Gamma)$ is a right Ω_*^{Spin} -module.
- If $M \in \Omega_*^{\text{Spin}}(B\Gamma)$ and $N \in \Omega_*^{\text{Spin}}$ then $M \times N$ has a natural Spin structure.

We have seen above, in (6.7), that there is a functor that forgets the Γ -structure. By forgetting the map to the classifying space of the fundamental group, one can define the forgetful functor $\Omega_*^{\text{Spin}}(B\Gamma) \rightarrow \Omega_*^{\text{Spin}}$. This forgetful functor can be split by associating to any element of Ω_*^{Spin} the trivial Principal Γ bundle $M \times \Gamma$. Let the reduced equivariant bordism groups $\tilde{\Omega}_*^{\text{Spin}}(B\Gamma)$ be the kernel of the forgetful functor. This leads to the decomposition

$$\Omega_*^{\text{Spin}}(B\Gamma) = \Omega_*^{\text{Spin}} \oplus \tilde{\Omega}_*^{\text{Spin}}(B\Gamma) . \quad (6.12)$$

The above can all be extended to Spin^c in a parallel way. One can also construct many classes of examples (see [78]). For instance, the ten-dimensional Spin^c cobordism group has rank $\text{Rank}_{\mathbb{Z}}(\Omega_{10}^{\text{Spin}^c}) = 4$.

Cobordism and positive scalar curvature. For any space M , $\Omega_n^{\text{Spin},+}(M)$ is the subgroup of $\Omega_n^{\text{Spin}}(M)$ defined by

$$\Omega_n^{\text{Spin},+}(M) = \{[N, f] : N \text{ is an } n\text{-dimensional Spin manifold with psc metric, } f : N \rightarrow M\} .$$

For a finitely presented group Γ , [63]

$$\text{Gromov - Lawson conjecture} \iff \Omega_n^{\text{Spin},+}(B\Gamma) = \ker(\text{per} \circ D) \quad (6.13)$$

$$\text{Gromov - Lawson - Rosenberg conjecture} \iff \Omega_n^{\text{Spin},+}(B\Gamma) = \ker \alpha . \quad (6.14)$$

Consider the example of cyclic fundamental group. Let X^{10} be a connected compact manifold with cyclic fundamental group. Assume that the universal cover \tilde{X}^{10} of X^{10} is Spin.

1. If X^{10} is Spin, then X^{10} admits a metric of positive scalar curvature if and only if $\hat{A}(X^{10}, \sigma)$ for all Spin structures on X^{10} [35] [105].
2. If X^{10} is orientable but not Spin, then X^{10} admits a metric of positive scalar curvature if and only if $\hat{A}(\tilde{X}^{10}) = 0$, where \tilde{X}^{10} is the universal cover of X^{10} [34] [106].

Consequences for the partition function. Let X^{10} be a Spin manifold of dimension $n \geq 5$ with fundamental group Γ , and let $u : X^{10} \rightarrow B\Gamma$ be the classifying map of the universal covering $\tilde{X}^{10} \rightarrow X^{10}$. Then, by the Gromov-Lawson-Rosenberg (GLR) conjecture, X^{10} has a positive scalar curvature metric iff the element $\alpha(X^{10}, u)$ in $KO(C_r^*(\Gamma))$ vanishes. Since the GLR conjecture is a theorem in our range of dimensions, then the spaces for which $\alpha = 0$, and hence for which there is no mod 2 anomaly in the type IIA partition function, are characterized; their α -invariant has as a source the subgroup $\Omega_n^{\text{Spin},+}(B\Gamma)$ of manifolds of positive scalar curvature admitting a map to the classifying spaces of their fundamental group.

6.2 Non-simply connected eleven-dimensional manifolds $\pi_1(Y^{11}) \neq 1$

Equivariant Spin and Spin^c cobordism. The discussion in eleven dimensions parallels that in ten dimensions from section 6.1. Let Γ be a spherical space form group and let $\varrho : \Gamma \rightarrow U(k)$ be fixed-point-free. Since we have a Spin condition: If $|\Gamma|$ is even then we suppose k to be even. $M = M(\Gamma, \varrho)$ has a Spin structure so that $M \in \tilde{\Omega}_*^{\text{Spin}}(B\Gamma)$. The above can all be extended to Spin^c in a parallel way.

1. If $M = S^m/\varrho(\Gamma)$ is Spin then $M \in \tilde{\Omega}_*^{\text{Spin}}(B\Gamma)$.
2. If $M = S^m/\varrho(\Gamma)$ is Spin^c then $M \in \tilde{\Omega}_*^{\text{Spin}^c}(B\Gamma)$.

There are many examples of such equivariant cobordism groups which are very far from being trivial. For example (see [78] for an extensive list), for cyclic groups

$$\Omega_{11}^{\text{Spin}^c}(B\mathbb{Z}_2) = \mathbb{Z}_{64} \oplus \mathbb{Z}_{16} \oplus 2 \cdot \mathbb{Z}_4 \oplus \mathbb{Z}_2, \quad \Omega_{11}^{\text{Spin}^c}(B\mathbb{Z}_3) = 2 \cdot \mathbb{Z}_{27} \oplus 2 \cdot \mathbb{Z}_9 \oplus 4 \cdot \mathbb{Z}_3, \quad (6.15)$$

and, for quaternionic groups,

$$\begin{aligned} \Omega_{11}^{\text{Spin}^c}(BQ_2) &= \mathbb{Z}_{128} \oplus \mathbb{Z}_{32} \oplus 4 \cdot \mathbb{Z}_8 \oplus 3 \cdot \mathbb{Z}_4 \oplus 7 \cdot \mathbb{Z}_2, \\ \Omega_{11}^{\text{Spin}^c}(BQ_3) &= \mathbb{Z}_{256} \oplus \mathbb{Z}_{64} \oplus 2 \cdot \mathbb{Z}_{16} \oplus 3 \cdot \mathbb{Z}_8 \oplus 3 \cdot \mathbb{Z}_4 \oplus 7 \cdot \mathbb{Z}_2. \end{aligned} \quad (6.16)$$

Thus, in extending Y^{11} to Z^{12} in the case of Spin^c with nontrivial fundamental group one encounters various potential obstructions. This implies that the study of the partition function in terms of index theory in twelve dimensions requires careful analysis in this case. We will not pursue that aspect further in this paper and leave it for future discussions.

Invariants from K-theory. For any finite group Γ , There is the standard isomorphism $R(\Gamma) \rightarrow K^0(B\Gamma)$, which assigns to every representation ϱ of Γ an associated bundle V_ϱ over $B\Gamma$ determined by the representation module associated to ϱ for every point of $B\Gamma$. By [12], there is a correspondence between the representation ring of the finite fundamental group $\Gamma = \pi_1(Y^{11})$ of an eleven-dimensional manifold Y^{11} bounding a twelve-manifold Z^{12} and the equivalence classes of vector bundles on Y^{11} , taken modulo the integers

$$R(\pi_1(Y^{11})) \rightarrow K^{-1}(Y^{11}; \mathbb{Q}/\mathbb{Z}). \quad (6.17)$$

This map sends the class of a representation ϱ to the class $[\varrho]$ of the representation vector bundle V_ϱ associated to ϱ over Y , and then using the pushforward in K-theory to obtain an element in the coefficient group of K-theory modulo the integers. This is done by means of the completion homomorphism, which

relates K_Γ^* and $K^*(B\Gamma)$, and the pullback of the corresponding classifying map $f : Y^{11} \rightarrow B\Gamma$. The inclusion of the trivial group in Γ induces a representation between representation rings. The kernel of this representation is the augmentation ideal I_Γ . The quotient $\tilde{R}(\Gamma) := R(\Gamma)/I_\Gamma$ is the reduced representation ring of Γ . The completion with respect to I_Γ is done by means of $c_{I_\Gamma} : I_\Gamma \rightarrow \tilde{K}^0(B\Gamma)$. The Bockstein homomorphism in K-theory associated to the short exact sequence of coefficients $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ is $\delta^{-1} : \tilde{K}^0(B\Gamma) \rightarrow K^{-1}(B\Gamma; \mathbb{Q}/\mathbb{Z})$. The composition

$$\gamma := \delta^{-1} \circ c_{I_\Gamma} : R(\Gamma)/I_\Gamma \rightarrow K^{-1}(B\Gamma; \mathbb{Q}/\mathbb{Z}) \quad (6.18)$$

sends the representation $\varrho : \Gamma \rightarrow U(k)$ of rank k to the class $\gamma(\varrho - k)$. Consider the pullback via f with respect to the $\pi_1(Y^{11})$ -action on Y^{11} ; we get a map $f^* : K^{-1}(B\Gamma; \mathbb{Z}) \rightarrow K^{-1}(Y^{11}; \mathbb{Q}/\mathbb{Z})$. The composition $f^* \circ \gamma : \tilde{R}(\Gamma) \rightarrow K^{-1}(Y^{11}; \mathbb{Q}/\mathbb{Z})$ sends the reduced representation $\varrho - k$ to a class $[\varrho]$ in $K^{-1}(Y^{11}; \mathbb{Q}/\mathbb{Z})$.

The invariant in the Spin^c case. When Y^{11} is Spin^c , the direct image map $K^{-1}(Y^{11}; \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$ sends the class of $[\varrho]$ in $K^{-1}(Y^{11}; \mathbb{Q}/\mathbb{Z})$ to a number in \mathbb{Q}/\mathbb{Z} . This number is the eta invariant of a corresponding Dirac operator twisted by the flat line bundle V_ϱ defined by the representation [12]. Below, we will take this flat line bundle to be associated to the fundamental group. Note that associated to every rank k is a unitary representation of Γ in some $U(k)$.

The eta-invariant associated to the fundamental group. Take ϱ to be a unitary representation of $\pi_1(Y)$ and let V_ϱ be the associated vector bundle $V_\varrho := (\tilde{Y}^{11} \times_\varrho \mathbb{R}^k) / \sim$, with $(\tilde{y}, v) \sim (g \cdot \tilde{y}, \varrho(g)v)$ for all $g \in \pi_1(Y^{11})$ (cf. (2.24)). Let D be a Dirac operator on a bundle E . The transition functions of V_ϱ are locally constant and the operator D extends naturally to an operator D_ϱ on $C^\infty(E \otimes V_\varrho)$. Define $\eta(D)(\varrho) := \eta(D_\varrho)$. The map $\varrho \rightarrow \eta(D)(\varrho)$ is additive and extends to a map from $RU(\pi_1(Y^{11}))$ to \mathbb{R} . Let $\tilde{\varrho} := \varrho(\lambda^{-1})^t$ be the dual virtual representation. The above works in any odd dimension.

Example 1: The circle S^1 . Let $n \geq 2$. Give S^1 the canonical \mathbb{Z}_ℓ structure and either of the two inequivalent Spin structures. Then (see [79])

$$\eta(S^1)(\varrho_0 - \varrho_s) = -\frac{s}{\ell} \in \mathbb{R}/\mathbb{Z}. \quad (6.19)$$

Interchanging the Spin structures replaces $\varrho_0 - \varrho_s$ by $\varrho_{\frac{\ell}{2}} - \varrho_{\frac{s+\ell}{2}}$ and leaves the eta invariant unchanged.

Example 2: General odd-dimensional spaces. Let D be the Dirac operator on a compact Spin manifold Y of odd dimension m which admits a metric of positive scalar curvature. Let ϱ be a unitary representation of $\pi_1(Y)$. Then $\ker(D_\varrho) = \{0\}$ and [62]

$$\begin{aligned} \eta(D_\varrho) &= \eta(D_{\tilde{\varrho}}) & \text{if } m \equiv 3 \pmod{4} \\ \eta(D_\varrho) &= -\eta(D_{\tilde{\varrho}}) & \text{if } m \equiv 1 \pmod{4}. \end{aligned} \quad (6.20)$$

Scalar curvature and geometric bordism groups. In order to include geometric data such as the metric, the fundamental group, and the Spin structure, one introduces geometric bordism groups. The geometric bordism groups $G^+ \Omega_{11}^{\text{Spin}}(B\Gamma, \xi)$ are defined as follows. Consider quadruples (Y^{11}, f, σ, g) , where f is a Γ structure on a closed manifold Y^{11} , where σ is a Spin structure on $TY^{11} \oplus f^* \xi$, and where g is a metric of positive scalar curvature on Y^{11} . Introduce the equivalence relation $[(Y^{11}, f, \sigma, g)] = 0$ if there exists a compact manifold Z^{12} with boundary Y^{11} so that the structures f and σ extend over Z^{12} and so that the metric g extends over Z^{12} as a metric of positive scalar curvature which is a product near the boundary. Assume that there exists a manifold Y_1^{11} which admits a metric g_1 of positive scalar curvature and which admits structures (f_1, σ_1) so that $[(Y^{11}, f, \sigma)] = [(Y_1^{11}, f_1, \sigma_1)]$ in $\Omega_{11}^{\text{Spin}}(B\Gamma, \xi)$. Then, by the results of [76] [124] [142] [143] [144], Y^{11} admits a metric of positive scalar curvature g so that $[(Y^{11}, f, \sigma, g)] = [(Y_1^{11}, f_1, \sigma_1, g_1)]$ in $G^+ \Omega_{11}^{\text{Spin}}(B\Gamma, \xi)$.

Let $[(Y^{11}, f, \sigma, g)] \in G^+ \Omega_{11}^{\text{Spin}}(B\Gamma; \xi)$ and let $\varrho \in RU_0(\Gamma)$. Assume that ξ is orientable so that σ_ξ is a Spin^c structure. Suppose that Y^{11} is the boundary of Z^{12} and that the structures extend from Y^{11} to Z^{12} . Then the APS index theorem [13] gives

$$\text{Ind}(D_\varrho) = \int_{Z^{12}} \dim \varrho \cdot \hat{A}(Z^{12}) + \eta(Y^{11}, \varrho). \quad (6.21)$$

Since Z^{12} has a metric of positive scalar curvature, then the kernel of the Dirac operator on Z^{12} with spectral boundary conditions is trivial, so that the index is zero. Therefore, the map which sends (Y^{11}, f, σ) to $\eta(Y^{11}, \varrho)$ extends to a homomorphism from $G^+ \Omega_{11}^{\text{Spin}}(B\Gamma, \xi)$ to \mathbb{R} (see [79]).

Example: Spherical space forms. We will use the discussion on Spin^c structures and representations from the end of section 3.3.2. Let δ be a linear representation so that $(\delta \det(\varrho))^{1/2}$ is a well-defined representation of Γ . Such a representation always exists (e.g. $\delta = \det(\varrho)$). Then M admits a Spin^c structure whose associated determinant line bundle is given by the representation δ . Give M a Spin^c structure with associated determinant line bundle given by the linear representation δ . Let $\varrho \in RU(\Gamma)$. If $2k - 1 \equiv 3 \pmod{4}$, assume $\varrho \in RU_0(\Gamma)$. Then [79]

$$\eta(M)(\varrho) = |\Gamma|^{-1} \sum_{g \in \Gamma, g \neq 1} \text{Tr}(\varrho(g)) \{ \delta(g) \det(\varrho(g)) \}^{1/2} \det(\varrho(g) - I)^{-1}. \quad (6.22)$$

A Spin^c example: The eta invariant of lens space bundles. [35] [79] Let $n \geq 2$ and let $\vec{a} := (a_1, \dots, a_5)$ be a collection of odd integers which are coprime to n . Let L_1, \dots, L_5 be complex line bundles over the Riemann sphere $S^2 = \mathbb{CP}^1$. Let $S(L_1 \oplus \dots \oplus L_5)$ be the associated sphere bundle of dimension eleven. This manifold is Spin iff the sum of the first Chern classes $c_1(L_1) + \dots + c_1(L_5)$ is even in $H^2(S^2; \mathbb{Z}) = \mathbb{Z}$. Assume this is the case. Let $\varrho_{\vec{a}}(\lambda)(\xi_1, \dots, \xi_5) := (\lambda^{a_1} \xi_1, \dots, \lambda^{a_5} \xi_5)$, for $\lambda \in \mathbb{Z}_n$, define a fiberwise action of \mathbb{Z}_n on $L_1 \oplus \dots \oplus L_5$. The restriction of this action defines a fixed point free action of \mathbb{Z}_n on $S(L_1 \oplus \dots \oplus L_5)$. Let the resulting quotient manifold be

$$Y^{11}(n; \vec{a}; L_1, \dots, L_5) := S(L_1 \oplus \dots \oplus L_5) / \varrho_{\vec{a}}(\mathbb{Z}_n). \quad (6.23)$$

Y^{11} admits a metric of positive scalar curvature. Furthermore, since $11 = 2k + 1$ with k odd, Y^{11} admits a Spin^c structure, with associated determinant line bundle defined by the representation ρ_1 . Define $\delta_{\vec{a}}$ to be $\frac{1}{2}(1 + a_1 + \dots + a_5)$. Let $\varrho \in RU_0(\mathbb{Z}_n)$. Then, using [79],

$$\eta(Y^{11})(\varrho) = -\frac{1}{2n} \sum_{\lambda \neq 1} \text{Tr}(\varrho(\lambda)) \lambda^{\delta_{\vec{a}}} \left\{ \prod_i \frac{1}{\lambda^{a_i} - 1} \right\} \left\{ \sum_i \int_{\mathbb{CP}^1} c_1(L_j) \frac{\lambda^{a_j+1}}{\lambda^{a_j} - 1} \right\}. \quad (6.24)$$

A Spin example: The eta invariant of lens space bundles. Consider the nine-dimensional manifold

$$X^9(n; \vec{a}; L_1, \dots, L_4) := S(L_1 \oplus \dots \oplus L_4) / \varrho_{\vec{a}}(\mathbb{Z}_n). \quad (6.25)$$

similarly to Y^{11} above. This X^9 admits a Spin structure. The formula for the eta invariant is similar to (6.24) with $\varrho \in RU(\mathbb{Z}_n)$ and $\delta_{\vec{a}} = \frac{1}{2}(a_1 + \dots + a_4)$. Note that in dimension seven, this is an $M^{p,q,r}$ manifold, i.e. a circle bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, which is important in flux compactification to gauged supergravity in four dimensions.

Trivial Spin structures: Framed eleven-dimensional manifolds and the Adams e -invariant. The Pontrjagin-Thom construction gives an identification of stable homotopy groups of spheres π^S with the cobordism groups of framed manifolds. A framing is a trivialization of the stable normal bundle. Up to homotopy, this is equivalent to a trivialization of the stable tangent bundle (because we are embedding in flat \mathbb{R}^m for very large m). A particular form of framing is a parallelism, which is a trivialization of the tangent

bundle. So, a parallelism \mathcal{P} on Y^{11} induces a framing on Y^{11} and hence defines an element $[Y^{11}, \mathcal{P}] \in \pi_{11}^S$. Since $\Omega_{11}^{\text{Spin}} = 0$ then $Y^{11} = \partial Z^{12}$ for some Spin twelve-manifold Z^{12} . The Spin structure induced on Y^{11} is the trivial Spin structure defined by \mathcal{P} . The Adams e -invariant for Y^{11} is a homomorphism $e : \pi_{11}^S \rightarrow \mathbb{Q}/\mathbb{Z}$ defined via the relative \hat{A} -invariant as (cf. [156])

$$e[Y^{11}, \mathcal{P}] = \frac{1}{2} \hat{A}(Z^{12}, Y^{11}), \quad (6.26)$$

in \mathbb{Q}/\mathbb{Z} and is independent of the choice of Z^{12} .

Let $g_{\mathcal{P}}$ be the Riemannian metric defined by \mathcal{P} , and consider $\bar{\eta}(g_{\mathcal{P}}) = \frac{1}{2} (h(g_{\mathcal{P}}) + \eta(g_{\mathcal{P}}))$. Then the analytic formula for the Adams e -invariant is obtained as follows [12]. Let ω^Z be a connection on Z^{12} extending the product connection defined by the connection ω^Y near Y^{11} . Let $p(\omega^Z)$ be the total Pontrjagin class on TZ^{12} defined by ω^Z . A closed one-form \mathcal{W} on the space of connections on Y^{11} is $\mathcal{W} = d\mathcal{F}$, where $\mathcal{F}(\omega^Z) = \int_{Z^{12}} \hat{A}(p(\omega^Z)) \bmod \mathbb{Z}$. Then

$$\begin{aligned} e[Y^{11}, \mathcal{P}] &= \frac{1}{2} \mathcal{F}(\mathcal{P}) = \frac{1}{2} [\mathcal{F}(g_{\mathcal{P}}) + (\mathcal{F}(\mathcal{P}) - \mathcal{F}(g_{\mathcal{P}}))] \\ &= \frac{1}{2} \left(\bar{\eta}(g_{\mathcal{P}}) - \int_{\mathcal{P}}^{g_{\mathcal{P}}} \mathcal{W} \right) \bmod \mathbb{Z}. \end{aligned} \quad (6.27)$$

In the case when $\int_{\mathcal{P}}^{g_{\mathcal{P}}} \mathcal{W} \in 2\mathbb{Z}$, the phase of the partition function (with no nontrivial bundles) will be given by the Adams e -invariant $e^{\pi i \bar{\eta}} = e^{2\pi i e[Y^{11}, \mathcal{P}]}$.

Let D_{ϱ} be the Dirac operator on Y^{11} twisted by a unitary representation ϱ of $\pi_1(Y^{11})$. Let $\bar{\eta}_{\varrho} = \bar{\eta}_{D_{\varrho}}$ and let $\underline{n} = \dim \varrho$ be the trivial n -dimensional representation for which $\bar{\eta}_{\underline{n}} = n\bar{\eta}_D$. Then $\tilde{\bar{\eta}}_{\varrho} = \bar{\eta}_{\varrho} - \bar{\eta}_{\underline{n}} \pmod{\mathbb{Z}}$ is independent of the metric on Y^{11} and is a cobordism invariant of (Y^{11}, ϱ) ; if $Y^{11} = \partial Z^{12}$ (as Spin^c manifolds) with ϱ extending a unitary representation of $\pi_1(Z^{12})$, then $\tilde{\bar{\eta}}_{\varrho} = 0$. Thus, the phase of the partition function in this case (with non-nontrivial bundles) is one.

Now consider a finite covering $\tilde{Y}^{11} \rightarrow Y^{11}$ of degree n with representation $\varrho : \pi_1(Y^{11}) \rightarrow O(n)$. Then

$$\begin{aligned} e[\tilde{Y}^{11}, \tilde{\mathcal{P}}] - ne[Y^{11}, \mathcal{P}] &= \frac{1}{2} (\bar{\eta}(\tilde{g}_{\tilde{\mathcal{P}}}) - n\bar{\eta}(g_{\mathcal{P}})) \bmod \mathbb{Z} \\ &= \frac{1}{2} \tilde{\bar{\eta}}_{\varrho} \bmod \mathbb{Z}. \end{aligned} \quad (6.28)$$

In this case, the phase will involve the terms on the left hand side, the second of which can be arranged to give a unit phase by appropriately choosing the degree of the covering when possible.

6.3 The second homotopy group and Spin^c structures

We have seen that the fundamental group provides a vast number of examples of Spin manifolds with possibly many Spin structures (see section 2.2). The main source of manifolds with Spin^c structures are manifolds with nontrivial second homotopy group. This is due to the fact that Spin^c structures are classified by the second integral cohomology group of the manifold (see section 3.1). If the space is 2-connected then, by the Hurewicz theorem, the first nonzero homotopy is the same as the homology in the same degree, and Spin^c structures exist on M if $H^2(M; \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z}_2)$ is surjective.

If the second homotopy group $\pi_2(M)$ is finite, then assuming further that the fundamental group $\pi_1(M)$

is trivial, Spin and Spin^c structures are equivalent. This is because of the diagram

$$\begin{array}{ccccccc}
 & & & \vdots & & & (6.29) \\
 & & & \downarrow & & & \\
 & & & \pi_2(M) & & & \\
 & & & \downarrow \partial & & & \\
 & & & \mathbb{Z}_2 \oplus \mathbb{Z} & & & \\
 & \nearrow \alpha \mapsto (0, \alpha) & & \downarrow & & & \\
 \mathbb{Z} \cong \pi_1(S^1) & \longrightarrow & \pi_1(Q \times P_1) & \longrightarrow & \pi_1(Q) & \longrightarrow & 1 \\
 & & \downarrow & & & & \\
 & & \pi_1(M) & = & 1 & &
 \end{array}$$

in which rows and arrows are exact [73]. Since $\pi_2(M)$ is finite, the image of ∂ is contained in the subgroup \mathbb{Z}_2 , so that either $\partial \equiv 0$ or $\text{im}(\partial) = \mathbb{Z}_2$. The first gives $\pi_1(Q) = \mathbb{Z}_2$ and Q admits a Spin^c structure. The second does not give a Spin^c structure on Q . For example, the five-manifold $M^5 = SU(3)/SO(3)$ has trivial π_1 and $\pi_2(M^5) = \mathbb{Z}_2$, and the homotopy exact sequence argument shows that $\pi_1(Q) = 1$, i.e. Q admits no Spin^c structure.

Sources for the fields from non-vanishing of π_1 and π_2 . Physical fields are generally taken to be cohomology classes. Hence such fields in a given degree can be supported on a manifold if the cohomology of the manifold is nonzero in that degree. For a connected manifold M , $\pi_1(M) \neq 0$ implies that $H_1(M) = \pi_1(M)$ and, by Poincaré duality, a field in $H^1(M)$ can be supported. Similarly for $\pi_2(M) \neq 0$, as we saw at the beginning of this section.

- Now taking $M = Y^{11}$, then the only fields in cohomological degree 1 or 2 should be related to E_8 gauge theory, since the only supergravity form field available is C_3 (with its curvature G_4) and are not of that degree.
- For $M = X^{10}$, there is no field strength of degree 1 but we can take flat Ramond-Ramond fields. For $H^2(X^{10}) \neq 0$, we have either a flat B -field or an RR field strength, essentially the curvature of the M-theory circle bundle. This latter will be considered in more detail in section 7.

We summarize the situation for the relation between low degree homotopy groups and the fields in ten and eleven dimensions in the following table.

Dimension	$\pi_1 \neq 0$	$\pi_2 \neq 0$
Ten	Flat RR 1-form connection C_1	Flat B-field B_2 and/or RR field F_2
Eleven	Flat connection for E_8 bundle	E_8 gauge field F

The case of $\pi_3 \neq 0$ is considered in [150].

Note on the fields in type IIB. We briefly consider the case of type IIB string theory. Here there is a 1-form, the RR field F_1 , which is supported by the first nontrivial homotopy or homology. Let X^{10} be a compact manifold and consider its associated Jacobian torus $J_X = H^1(X^{10}; \mathbb{R})/H^1(X^{10}; \mathbb{Z})$. Let \tilde{X} denote the universal covering of X^{10} . Define a family of flat complex line bundles L over X^{10} parametrized by J_X by taking the quotient, as in [107],

$$L := \left(\tilde{X} \times H^1(X^{10}; \mathbb{R}) \times \mathbb{C} \right) / \pi_1(X^{10}) \times H^1(X^{10}; \mathbb{Z}), \quad (6.30)$$

where the action of $\pi_1(X^{10}) \times H^1(X^{10}; \mathbb{Z})$ is given by $\phi_{(g,h)}(\tilde{x}, v, z) = (\tilde{g}x, v + h, e^{2\pi i v(g)z})$. We can in this case also consider L as a twist for spinor modules $S \otimes L = (S^+ \otimes L) \oplus (S^- \otimes L)$, as a pair of vector bundles over X^{10} parametrized by J_X . We can also extend the flat connection on L to define a Dirac operator $D_v^+ : \Gamma(S^+ \otimes L_v) \rightarrow \Gamma(S^- \otimes L_v)$ for $v \in J_X$. Of course we could also do the same for type IIA string theory, using the fields in the above table.

7 Families, Eta Forms and Harmonic Forms

7.1 The B -field and harmonic representatives for the Spin^c structure

The C -field in eleven dimensions as a harmonic representative of the String structure is described in [150]. Here we consider the case of the B -field in ten-dimensional type IIA string theory.

The Euler class and the harmonic representative of the Ramond-Ramond 2-form. Let \mathcal{L} be a complex line bundle over X^{10} equipped with a smooth fiber metric and a Riemannian connection ∇ . Let $S(\mathcal{L}) := \{\xi \in \mathcal{L} : |\xi| = 1\}$ be the associated circle bundle. S^1 acts transitively on the fibers of $S(\mathcal{L})$ by complex multiplication, so $S(\mathcal{L})$ is a principal circle bundle. Conversely, every principal bundle arises this way. The Chern form of a principal circle bundle over X^{10} is an invariantly defined real closed two-form on X^{10} given by $c_1(\nabla) := \mathcal{F}(\nabla)$. The de Rham cohomology class is independent of the connection chosen and represents the complexification of c_1 . This is the description of the Ramond-Ramond (RR) 2-form F_2 , in the absence of a cosmological constant (i.e. the RR 0-form F_0).

We could ask for a harmonic representative for F_2 . Indeed, there exists a unitary connection ∇ on \mathcal{L} so that the curvature \mathcal{F} is harmonic. This can be shown as follows (see e.g. [79]). Let ∇' be any unitary connection on \mathcal{L} with curvature two-form \mathcal{F}' . Since this is a closed two-form, the Hodge-de Rham theorem ensures the existence of a harmonic two-form $\mathcal{F}_{\text{harm}}$ with $[\mathcal{F}'] = [\mathcal{F}_{\text{harm}}]$ in cohomology, so that $\mathcal{F}_{\text{harm}} = \mathcal{F}' + dv$, where v is a smooth 1-form on X^{10} . A unitary connection ∇ can be built out of ∇' and v as $\nabla := \nabla' + iv$, so that

$$\mathcal{F} = dA' + dv = \mathcal{F}' + dv = \mathcal{F}_{\text{harm}}. \quad (7.1)$$

Next we ask: When is this harmonic representative nontrivial? Since the first Chern class identifies $\text{Vect}_{\mathbb{C}}^1(X^{10})$ with $H^2(X^{10}; \mathbb{Z})$ and since $0 \neq H^2(X^{10}; \mathbb{R}) = H^2(X^{10}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ we may choose $\mathcal{L} \in \text{Vect}_{\mathbb{C}}^1(X^{10})$ so that $0 \neq c_1(\mathcal{L})$ in $H^2(X^{10}; \mathbb{R})$. Therefore, if $H^2(X^{10}; \mathbb{R}) \neq 0$, then there exists a complex line bundle \mathcal{L} over X^{10} and a unitary connection on \mathcal{L} so that the curvature \mathcal{F} is harmonic and nontrivial.

The B -field as a harmonic representative of the Spin^c structure. Next we consider the B -field. We can use the line bundle corresponding to the M-theory circle to be the line bundle which enters in defining B as a connection on a gerbe. This brings out the relation between B and the RR field F_2 . This is most manifest in the presence of the RR 0-form (the cosmological constant), where $dF_2 = F_0 H_3$, so that essentially $F_2 = F_0 B_2$. Let $P = P_{SO}(X^{10})$ be a principal $SO(10)$ bundle with connection ω_X over a 10-dimensional Riemannian manifold (X^{10}, g_X) . Consider a natural one-parameter family of metrics g_t on P , $g_t = g_X + tg_{SO}$. The adiabatic limit is given by taking $t \rightarrow 0$ (see more in section 7.3). We use the following cohomological definition of Spin^c structure. A Spin^c structure on an $SO(n)$ bundle $P \rightarrow M$ is a cohomology class $\sigma_X^c \in H^2(P; \mathbb{Z})$ such that $i^* \sigma_X^c \neq 0 \in H^2(SO(n); \mathbb{Z}) \cong \mathbb{Z}_2$, where $i : SO(n) \hookrightarrow P$ is the fiberwise inclusion. A Spin^c structure, topologically, is a choice of a particular complex line bundle $L \rightarrow P$ characterized by its first Chern class $\sigma_X^c \in H^2(P; \mathbb{Z})$. To any Spin^c structure there is an associated line bundle \mathcal{L} on X^{10} . This is determined by requiring that $\pi^* c_1(\mathcal{L}) = 2\sigma_X^c \in H^2(P; \mathbb{Z})$. However, upon introducing a harmonic metric g_X on X^{10} and a connection on P (which is usually determined by g), then taking the harmonic representative of σ_X^c in the adiabatic limit produces a canonical 2-form $B_{g, \sigma_X^c} \in \Omega^2(X^{10})$, such that

$$\pi^* c_1(\mathcal{L}) = 2[B] \in H^2(P; \mathbb{R}). \quad (7.2)$$

This gives that $2B_{g,\sigma_X^c} \in \Omega^2(X^{10})$ is the curvature of some connection on \mathcal{L} . Furthermore, $B_{g,c}$ is harmonic so it picks out a class of connections on \mathcal{L} with energy-minimizing curvature. The harmonic representative in the adiabatic limit of (the real cohomology class given by) σ_X^c is the form equal to the pullback of a 2-form from X^{10} , using [141],

$$\lim_{t \rightarrow 0} [\sigma_X^c]_{gt} = \pi^* B_{g,\omega_X,\sigma_X^c} \in \Omega^2(P), \quad (7.3)$$

where the 2-form $B_{g,\omega_X,\sigma_X^c} \in \Omega^2(X^{10})$. Furthermore, if the Spin^c structure is changed by $\xi \in H^2(X; \mathbb{Z})$ then

$$[\sigma_X^c + \pi^* \xi]_0 = [\sigma_X^c]_0 + \pi^* [\xi]_g. \quad (7.4)$$

For $\xi \in H^2(X^{10}; \mathbb{Z})$ the adiabatic harmonic representative of the class pulled back to P is just the pullback of the harmonic representative on X^{10} :

$$[\pi^* \xi]_0 = \pi^* [\xi]_g \in \Omega^2(P). \quad (7.5)$$

7.2 Families of Dirac operators and superconnections over X^{10}

7.2.1 Families of Dirac operators on the M-theory disk

Consider the two-dimensional disk \mathbb{D}^2 (which is compact but not closed), with a collection of Dirac operators $\{D_x\}$, parametrized by points of the ten-dimensional space X^{10} . As x varies continuously in X^{10} , the index $\text{Ind}(D_x)$ remains constant. However, we have a family of Dirac operators which can be nontrivial globally, in an analogous way that a vector bundle, which is locally a product of a vector space with the base space, can be globally nontrivial. Let $\pi_D : Z^{12} \rightarrow X^{10}$ be a family of disks over X^{10} . A family of Dirac operators on this family of disks consists of:

- (i) A \mathbb{Z}_2 -graded Hermitian vector bundle E over Z^{12} .
- (ii) For each $x \in X^{10}$, a Dirac operator D_x on the manifold $Z_x = \pi_D^{-1}(x)$, acting on section of the restriction of the bundle E to Z_x , such that:
- (iii) The operators D_x vary smoothly with x , i.e. they are restrictions to \mathbb{D}_x^2 of a single Dirac operator on Z^{12} .

A family of Dirac operators over X^{10} can also be defined as the leafwise Dirac operator on the following groupoid (see [91]). A family G_{π_D} of pair groupoids parametrized by X^{10} is the smooth groupoid given by:

- The object space is Z^{12} ;
- The morphism space is $\{(z_1, z_2) \in Z^{12} \times Z^{12} : \pi_D(z_1) = \pi_D(z_2) \in X^{10}\}$;
- The source and the range maps are $s(z_1, z_2) = z_1, r(z_1, z_2) = z_2$;
- The composition law is $(z_1, z_2) \cdot (z_2, z_3) = (z_1, z_3)$;
- The inverse is $(z_1, z_2)^{-1} = (z_2, z_1)$;
- The inclusion of identities is $z \mapsto (z, z)$.

This is equivalent to a family of Dirac operators in the sense that the C^* -algebra of this groupoid G_{π_D} is Morita equivalent to $C(X^{10})$, the algebra of continuous functions on X^{10} . The family index gives rise to an element of the K-theory of X^{10} , via the isomorphism $K^0(X^{10}) = K(C_r^*(G_{\pi_D}))$ with the K-theory of the groupoid C^* -algebra (cf. section 6.1). The vertical tangent bundle to the fibers is

$$T_{\pi_D} Z^{12} = \ker(\pi_{D*} : TZ^{12} \rightarrow TX^{10}). \quad (7.6)$$

Equipping this family with a Spin^c structure gives a family of Dirac operators on $Z^{12} \rightarrow X^{10}$. Then there is a map $\pi_{D!} : K^0(Z^{12}) \rightarrow K^0(X^{10})$ associated to the K -oriented map π_D .

There are two forms of the index theorem: K-theoretic and cohomological. The first is more powerful, since the Chern character, mapping K-theory to cohomology, loses torsion information. The cohomological formula is

$$\text{ch}(\text{Ind} D) = \int_{\mathbb{D}^2} \text{ch}(\sigma_D) \text{Todd}(T_{\pi_D} Z^{12} \otimes \mathbb{C}) \in H^*(X^{10}), \quad (7.7)$$

where the integral is integration over the fiber, going from $H^*(Z^{12})$ to $H^*(X^{10})$. We will consider disk bundles further in section 7.4.

Ramond-Ramond (RR) fields via Dirac families. Consider Dirac operators with coefficients in a bundle $E \rightarrow \mathbb{D}^2$ for a family $\mathbb{D}^2 \rightarrow X^{10}$. A geometric form of the index theorem gives the index of the family of Dirac operators is the image of E under pushforward in a form of K-theory. When E is complex then complex K-theory pushforward gives, via $K(\mathbb{D}^2) \rightarrow K(X^{10})$, an element of the K-theory of the base. The Dirac operator D_x is taken to be a field parametrized by points $x \in X^{10}$. Take a complex vector bundle E to be $\ker D_x$ and its conjugate \overline{E} to be $\text{coker } D_x$. Then

$$E - \overline{E} = \ker D_x - \text{coker } D_x , \quad (7.8)$$

which is the index of D_x , represents an element of K-theory $K(X^{10})$. The RR fields satisfy the quantization condition [125] [69]

$$F(E) = \sqrt{\widehat{A}(X^{10}) \text{ch}(E)} \quad (7.9)$$

corresponding to a complex vector bundle E (or the class of such a bundle). Thus, for the tensor product $E \otimes E'$, with another vector bundle E' , we get

$$\begin{aligned} F(E) \wedge F(E') &= \widehat{A}(X^{10}) \text{ch}(E) \text{ch}(E') \\ &= \widehat{A}(X^{10}) \text{ch}(E \otimes E') \\ &= \text{Ind}_{E \otimes E'}(X^{10}) . \end{aligned} \quad (7.10)$$

The mod 2 index. If we take $E' = \overline{E}$ to be the complex conjugate of E , then we get the tensor bundle $E \otimes \overline{E}$, which is a real bundle. Thus, in this case, the complex K-theory description refines to that of KO-theory $KO(X^{10})$.

7.2.2 Families of Dirac operators on the M-theory circle

In this section we consider the pushforward of bundles in eleven dimensions to bundles in ten (and lower) dimensions at the level of K-theory. Several new features arise, including the appearance of infinite-dimensional bundles. Let $\pi : Y^{11} \rightarrow X^{10}$ be a smooth fibration of manifolds with a Spin structure along the fibers, so that X^{10} is Spin with Spin bundle $SX \rightarrow X^{10}$. Form a metric along the fibers, i.e., a metric on $T(Y^{11}/X^{10})$, and a smoothly varying family of “horizontal subspaces” transverse to $\ker \pi_*$.

- A family of circles over a ten-dimensional manifold $(Y_x^{11} \mid x \in X^{10})$ is a family of fibers $Y_x^{11} = \pi^{-1}(x)$ of a smooth circle bundle $\pi : Y^{11} \rightarrow X^{10}$. Let $T(Y^{11}/X^{10}) = TS^1 \subset TY^{11}$ be the bundle of vertical tangent vectors with dual $T^*(Y^{11}/X^{10}) \cong T^*Y^{11}/\pi^*T^*X^{10}$.
- A family of vector bundles $(E_x \mid x \in X^{10})$ is a smooth vector bundle $E \rightarrow Y^{11}$, so that E_x is the restriction of the bundle E to Y_x^{11} . To the family $E \rightarrow Y^{11}$ we associated the infinite-dimensional bundle π_*E over X^{10} , whose fiber $(\pi_*E)_x$ at $x \in X^{10}$ is the Fréchet space $\Gamma(Y_x^{11}, E_x)$. A smooth section of π_*E over X^{10} is defined to be a smooth section of E over Y^{11} :

$$\Gamma(X^{10}, \pi_*E) = \Gamma(Y^{11}, E) . \quad (7.11)$$

Then the geometric family of Dirac operators is defined as a differential geometric version of the analytical index

$$\text{Ind } D = \ker D_x - \text{coker } D_x , \quad (7.12)$$

which is the formal difference of parametrized families of vector spaces. If $\dim(\ker D_x)$ is constant in x , then each term determines a vector bundle and so $\text{Ind } D$ makes sense as an element of $K(X^{10})$. Similarly, the discussion can be extended to Dirac operators D_E twisted by the bundle E .

The tangent bundles and Killing vector. Let X^{10} be a ten-dimensional compact connected Spin manifold with a fixed Spin structure. Let g_X be a metric on TX^{10} , let ∇^X be the associated Levi-Civita connection and R_X the curvature of ∇^X . Let $Z^{12} \rightarrow X^{10}$ be a 2-dimensional oriented vector bundle over X^{10} . Let g_Z be a metric on Z^{12} and let ∇^Z be a connection on Z^{12} preserving g_Z and with curvature R_Z . Let Y^{11} be the boundary of Z^{12} with metric g_Y , so that $TY^{11} = TS^1 \oplus \pi^*TX^{10}$, $g_Y = g_Z|_{Y^{11}} = g_{S^1} \oplus \pi^*g_X$. Let ∇^{LC} denote the Levi-Civita connection of g_Y . Then Y^{11} is a circle bundle over X^{10} with structure group $SO(2)$ acting by isometries on the fibers. Furthermore, it carries a canonical induced Spin structure induced from the Spin structure on X^{10} . A Spin structure is also induced on the fiber (see section 2.5).

Consider the free circle action on Y^{11} and let ξ denote the Killing vector generating this action. Locally, $\xi = d\theta$, where θ is the local coordinate on S^1 . For every $y \in Y^{11}$, the tangent space to Y^{11} at y splits into vertical and tangent horizontal spaces $T_y Y^{11} = TS_y^1 \oplus T^H Y_y^{11}$, where TS_y^1 is the one-dimensional subspace spanned by $\xi(y)$ and $T^H Y_y^{11} = (TS_y^1)^\perp$ is the orthogonal complement. The projection π_* defines an isomorphism $T^H Y_y^{11} \cong T_{\pi y} X^{10}$ and there is a unique metric h_X on X^{10} for which this is also an isometry. The horizontal distribution $T^H Y^{11}$ defines a one-form ξ^* such that $\ker(\xi^*) = T^H Y^{11}$ and is normalized so that $\xi^*(\xi) = 1$. Locally, $\xi^* = d\theta + A$, where A is a horizontal one-form, the RR one-form potential. Its field strength F_2 pulls back to the curvature $d\xi^*$ of the principal connection. F_2 is both horizontal and invariant, hence basic (see [68] [116]).

Geometric representative of the Euler class of the circle bundle. Let π^{S^1} be the orthogonal projection on TS^1 with respect to g_Y . Let ∇ be a connection on TY^{11} defined as (cf. [29])

$$\begin{aligned} \nabla_z v_z &= \pi^{S^1}(\nabla_z^{LC} v_z), & \nabla_\mu v_z &= \pi^{S^1}(\nabla_\mu^{LC} v_z), \\ \nabla_z v_\mu &= 0, & \nabla_\mu v_\nu &= \nabla_\mu^X v_\nu, \end{aligned} \quad (7.13)$$

for v a vector with component z along the eleventh direction or μ along X^{10} . As indicated in the introduction, this is more general than the usual Kaluza-Klein and Scherk-Schwarz settings for dimensional reduction in the sense of allowing $\nabla_\mu v_z$ to be nonzero in general. We define the following:

1. S , a tensor defined by $S = \nabla^{LC} - \nabla$. Then [169] $S(\xi)(\xi) = 0$ for $\xi \in TS^1$ the unit vector field determined by g_Y and the Spin structure on TS^1 .
2. T , the torsion of ∇ defined by $T_{\mu\nu} = -S_{\mu\nu} + S_{\nu\mu}$.

As above, let $\xi^* \in T^*S^1$ be the dual of ξ . Then, since ∇^{LC} is torsion-free, we have the pairing

$$\langle T(U, V), \xi \rangle = d\xi^*(U, V), \quad U, V \in TX^{10}. \quad (7.14)$$

Therefore, T determines a 2-form in $\Lambda^2(T^*X^{10})$ such that $\frac{1}{2\pi}T$ represents the Euler class $e(Z^{12})$ of Z^{12} . Thus the above torsion can be viewed as the RR 2-form F_2 . Let $\{e_1, \dots, e_{10}\}$ be an orthonormal basis of TX^{10} and $\{e_1^*, \dots, e_{10}^*\}$ be the dual basis, i.e. a basis of T^*X^{10} . Set

$$c(T) = \frac{1}{2} \sum_{a,b} e_a^* e_b^* c(T(e_a, e_b)). \quad (7.15)$$

We view this as the $\gamma^{\mu\nu} F_{\mu\nu}$ (in the physicist's component notation) term in the dilatino supersymmetry transformation. We will make more connection below.

The Killing spinors. Consider the case when the circle action in M-theory lifts to an action on the Spin bundle. This is infinitesimally generated by the spinorial Lie derivative

$$\mathcal{L}_\xi \varepsilon = \nabla_\xi \varepsilon + \frac{1}{4} c(d\xi^*) \varepsilon, \quad (7.16)$$

for any spinor ε (compare to expression (2.32)). An invariant Killing spinor, specified by $D_\xi \varepsilon = 0$ and $\mathcal{L}_\xi \varepsilon = 0$, gives rise to a IIA Killing spinor [66]. In the presence of the dilaton ϕ , i.e. for non-constant length vector, we have that expression (7.16) is changed to

$$(\mathcal{L}_\xi - \nabla_\xi) \varepsilon = \frac{1}{4} e^{-\phi/2} c(d\xi^*) \varepsilon, \quad (7.17)$$

which represents the supersymmetry transformation of the dilatino in type IIA superstring theory.

Quillen superconnection and differential geometric representatives. Quillen's superconnection [140] is the differential geometric refinement of an element in K-theory in the same way that a connection on a smooth vector bundle is the differential geometric refinement of an equivalence of topological vector bundles. Let $V = V^+ \oplus V^-$ be a smooth \mathbb{Z}_2 -graded vector bundle over a smooth ten-dimensional manifold X^{10} . A superconnection on V consists of an ordinary connection $\nabla = \nabla^+ \oplus \nabla^-$ and a linear endomorphism $L : V \rightarrow V$, which anticommutes with the grading, i.e. $L(V^\pm) \subseteq V^\mp$. The superconnection is the operator $\nabla + L$ acting on the space $\Omega^*(X^{10}, V)$ of differential forms on X^{10} with values in the bundle V . The K-theory element corresponding to $\nabla + L$ is $V^+ \xrightarrow{L} V^-$. Define the supertrace as the map $\text{Tr}_s : \Omega^*(X^{10}, \text{End} V) \rightarrow \Omega^*(X^{10})$ taking $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ to $\text{tr } \alpha - \text{tr } \delta$. Then the curvature is $(\nabla + L)^2$ and the Chern character is $\text{Tr}_s \exp(\nabla + L)^2$, and the latter differential form represents the topological Chern character of the associated K-theory element [140].

Bismut superconnection. The Bismut superconnection is obtained by applying Quillen's construction to a geometric family of Dirac operators. In our setting we have the following family $\pi : Y^{11} \rightarrow X^{10}$. The corresponding construction gives an *infinite-dimensional* superconnection over X^{10} . The fiber of the vector bundle at $x \in X^{10}$ is the space of spinors on Y_x^{11} , the fiber of π at x . The metric on Y_x^{11} gives this space an L^2 inner product, and the horizontal subspaces give it a unitary connection. The operator L is the Dirac operator D modified by a degree 2 term which is essentially the curvature T of the field of horizontal planes (7.14). The superconnection represents the associated differential geometric shriek map $\pi_!$, the pushforward in K-theory. Because D is Fredholm there is an element of $K(X^{10})$ associated to the superconnection which is precisely $\text{Ind} D$ in (7.12). Let $\tilde{\nabla}$ be the natural lifting of ∇ to the infinite-dimensional vector bundle $\Gamma(S(S^1) \otimes \mathbb{R}(TY^{11}|_{S^1}))$ over X^{10} , where \mathbb{R} denotes symmetric or antisymmetric powers. The connection $\tilde{\nabla}$ is unitary. The Bismut superconnection for the family $\{D_{S^1, \mathbb{R}}\}$ is [29]

$$\mathbb{A}_u = \tilde{\nabla} + u D_{S^1, \mathbb{R}} - \frac{c(T)}{4u}. \quad (7.18)$$

The last term, which is the difference of the Bismut superconnection and the Quillen superconnection, is the Clifford multiplication with the horizontal distribution, which can be considered as a two-form on X^{10} with values in the vertical vector fields.

The interpretation: Expression (7.18), which takes the important effect of the family on the M-theory circle into account, is a families version of the dilatino supersymmetry transformation (7.17), where the Bismut superconnection replaces the spinorial Lie derivative \mathcal{L}_ξ and the Quillen superconnection replaces the covariant derivative on spinors ∇_ξ .

Now consider the Bismut superconnection (7.18) in the limit $u \rightarrow 0$. In this case,

- The last term is dominant so that the effect of the torsion field takes over.
- The adiabatic limit is essentially taking $u \rightarrow 0$ in the Quillen superconnection, i.e. in the first two terms in (7.18).
- Comparing with the expression of the Dirac operator (2.38) we see that u is essentially $e^{-\phi}$, with ϕ the dilaton. In dynamical terms, e^ϕ represents the string coupling constant, so that u is the inverse

of the string coupling constant $e^{-\phi}$ (up to possible normalization factors). Therefore, the $u \rightarrow 0$ limit is the weak string coupling limit. Viewed from a different angle, this corresponds to $\phi \rightarrow \infty$, which geometrically means that the volume of the fiber is getting large.

- The behavior as $u \rightarrow 0$ leads to a differential geometric Riemann-Roch formula. Indeed, the Chern character of the Bismut superconnection approaches [29]

$$\int_{Y/X} \widehat{A}(\Omega^{(Y/X)}) \quad (7.19)$$

as $u \rightarrow 0$. Here $\Omega^{(Y/X)}$ is the curvature of the connection on $T(Y/X)$ determined by the geometric data. The differential form in (7.19) represents in de Rham cohomology the cohomology class in $\text{ch}(\text{Ind } D) = \pi_*(\widehat{A}(Y/X))$.

7.3 Equivariant eta invariants, eta forms, and adiabatic limits

The Spin bundle. Now that we described the integration over the fiber, we can continue the discussion from section 7.2.2 and generalize to Clifford modules. Let \mathcal{E} be a Clifford module along the fibers of π (e.g. an E_8 bundle). This means that we have a Hermitian vector bundle over Y^{11} with a skew-adjoint action $c : C(T^*(S^1)) \rightarrow \text{End}(\mathcal{E})$ of the vertical Clifford bundle of π , and a Hermitian connection $\nabla^{\mathcal{E}}$ compatible with this action, $[\nabla_Y, c(\xi_z)] = c(\nabla_Y^{\mathcal{E}} \xi_z)$, for $Y \in \Gamma(Y^{11}, TY^{11})$ and $\xi_z \in \Gamma(Y^{11}, T^*S^1)$. There is a one-to-one correspondence (see e.g. [27])

$$\{\text{Superconnections on the bundle } \pi_*\mathcal{E}\} = \{\text{Dirac operators on the Clifford module } SX^{10} \otimes \pi_*\mathcal{E}\}, \quad (7.20)$$

that is, an isomorphism

$$\Gamma(X^{10}, SX^{10} \otimes \pi_*\mathcal{E}) \cong \Gamma(Y^{11}, \pi^* SX^{10} \otimes \mathcal{E}). \quad (7.21)$$

The adiabatic limit for circle bundles. Consider S^1 -equivariant Dirac operator D on $\mathcal{E} \rightarrow S^1$. Let \overline{D} on $\overline{\mathcal{E}}$ be a family of Dirac operators on X^{10} induced by D . Then $\mathcal{E}_Y := \overline{\mathcal{E}} \otimes \pi^* SX^{10}$ becomes a Clifford module over Y^{11} and $\nabla^{\overline{\mathcal{E}}}$ induces a Clifford connection on \mathcal{E}_Y . Rescale the metric on the fibers by $t > 0$. Then the Dirac operator $D_{Y,t}$ may be viewed as the Dirac operator on $\pi_*\mathcal{E}_Y = \pi_*\overline{\mathcal{E}} \otimes SX^{10}$ associated to the Bismut Levi-Civita connection \mathbb{A}_t on $\pi_*\overline{\mathcal{E}}$. The results of Bismut-Cheeger [30] and Dai [43] give

$$\lim_{t \rightarrow 0} \eta(D_{Y,t}) \equiv 2(2\pi i)^{-\frac{\dim X}{2}} \int_X \widehat{A}(X) \wedge \widehat{\eta}(\overline{D}) \quad \text{mod } \mathbb{Z}, \quad (7.22)$$

where $\widehat{\eta}$ is an even form on X^{10} defined by Bismut-Cheeger [30]

$$\widehat{\eta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \text{Tr}^{\text{even}} \left[\left(D_{S^1, \mathbb{R}} + \frac{c(T)}{4u} \right) \exp(-\mathbb{A}_u^2) \right] \frac{du}{2\sqrt{u}}. \quad (7.23)$$

1. Since $u \sim e^{-\phi}$ then $u = 0$ corresponds to $\phi = +\infty$ and $u = \infty$ to $\phi = -\infty$.
2. The integral is then over all ϕ and resembles a soliton configuration. Thus we are summing over all dilaton configurations, that is, in a sense, integrating over string coupling parameters.

Eta forms on circle bundles giving an H -field on the base. Consider the S^1 bundle $\pi : Y^{11} \rightarrow X^{10}$, with the E_8 vector bundle V over Y^{11} . Then, considering the degree 2 eta-form,

$$d\widehat{\eta}_2 = \int_{S^1} a \in \Omega^3(X^{10}). \quad (7.24)$$

As considered in [148], this gives the B -field and the H -field in type IIA string theory as the differential forms $\widehat{\eta}_2$ and $d\widehat{\eta}_2$, respectively. Refinement to Deligne cohomology will be considered in section 7.4.

Adiabatic limit for bundles with higher-dimensional fibers. Consider a fiber bundle $M^d \rightarrow Y^{11} \xrightarrow{\pi_M} X^{11-d}$ with a Spin structure on the vertical tangent bundle $TM^d = \ker(d\pi_M)$, with curvature R_{TM} . Let V be a vector bundle on Y^{11} (which will be either an E_8 bundle or the Rarita-Schwinger bundle, or some formal combination) with a connection and curvature F^V . The differential of the eta-form will define a differential form over the base X^{11-d} obtained by integrating an index formula over the fiber M^d ,

$$d\hat{\eta} = \int_{M^d} \hat{A}(R_{TM}) \wedge \text{ch}(F^V). \quad (7.25)$$

Example: S^3 -bundles. Consider the following example. Let $M^d = S^3$. Taking V to be the E_8 bundle with characteristic class a of degree 4, then, with $\hat{A}(R_{TM}) = 1$, we have that we get a degree zero eta-form on the base X^8 obtained by integrating a over S^3 (up to a sign)

$$d\hat{\eta}_0 = \int_{S^3} a. \quad (7.26)$$

Thus, we get an eta-function $\eta = \hat{\eta}_0$ over M^7 , the boundary of the space X^8 ,

$$\begin{aligned} S^3 \longrightarrow Y^{11} &\longrightarrow X^8 \longleftarrow M^7 \\ a &\longmapsto d\eta \longleftarrow \eta. \end{aligned} \quad (7.27)$$

Such a space is guaranteed to exist because of the vanishing of the Spin cobordism group in dimension seven, $\Omega_7^{\text{Spin}} = 0$. Also, if we extend an E_8 bundle then we would need $\Omega_7^{\text{Spin}}(K(\mathbb{Z}, 4))$, which indeed vanishes (see e.g. [95]). The case of the Rarita-Schwinger bundle is similar.

Now let $\pi : Y^{11} \rightarrow X$ be a totally geodesic fiber bundle with odd-dimensional fiber M and structure group G . Similar arguments give (7.22). If the dimension of the fiber is even then the infinite-dimensional bundle $\pi_* E$ is graded, i.e. it splits as $\pi_* E^+ \oplus \pi_* E^-$. This case will be considered in section 7.4.

Equivariant eta invariants. Let G act isometrically on Y^{11} , a compact oriented Riemannian 11-manifold. Let D be a G -equivariant Dirac operator on a G -equivariant Dirac bundle \mathcal{E} over Y^{11} . One can form equivariant eta invariants with respect to either the group or its Lie algebra. The latter is the \mathfrak{g} -equivariant eta invariant $\eta_{\mathbf{X}}(D)$ with respect to an element $\mathbf{X} \in \mathfrak{g}$ of the Lie algebra \mathfrak{g} [81]. The former is the G -equivariant eta invariant $\eta_{e^{-\mathbf{X}}}(D)$ with respect to an element $e^{-\mathbf{X}}$ of the Lie group G [59]. The two are directly related; in fact, the difference $\eta_{\mathbf{X}}(D) - \eta_{e^{-\mathbf{X}}}(D)$ is locally computable when Y^{11} is a boundary, a fact that allows for relating G -equivariant η -invariants and η -forms of totally geodesic fiber bundles with structure group G [81].

Let $\mathbb{C}[\mathfrak{g}^*]$ denote the space of formal power series in $\mathbf{X} \in \mathfrak{g}$. Goette [81] defines equivariant eta invariants $\eta_{\mathbf{X}}(D) \in \mathbb{C}[\mathfrak{g}^*]$ such that when $\mathbf{X} = \Omega$ this gives the eta-form of any family of Dirac operators associated to a G -principal bundle over X with curvature $\Omega \in \mathcal{A}^2(X; \mathfrak{g}) := \Gamma(\Lambda^2 X) \otimes \mathfrak{g}$, a \mathfrak{g} -valued 2-form on X . For $\mathbf{X} \in \mathfrak{g}$, let $\mathbf{X}_Y(y) = \frac{d}{dt}|_{t=0} e^{-t\mathbf{X}} y$ be the Killing field induced by \mathbf{X} , let $c_{\mathbf{X}}$ denote Clifford multiplication with \mathbf{X}_Y , and let $\mathcal{L}_{\mathfrak{g}}$ denote the infinitesimal action of \mathfrak{g} on \mathcal{E} . Define the \mathfrak{g} -equivariant Dirac operator $D_{\mathbf{X}} := D - \frac{1}{2} c_{\mathbf{X}}$ and the equivariant Bismut Laplacian (cf. [27])

$$\mathcal{H}_{\mathbf{X}} := D_{-\mathbf{X}}^2 + \mathcal{L}_{\mathfrak{g}} = (D + \frac{1}{2} c_{\mathbf{X}})^2 + \mathcal{L}_{\mathfrak{g}}. \quad (7.28)$$

This is a generalized Laplacian which depends polynomially on $\mathbf{X} \in \mathfrak{g}$ and can be thought of as a quantum analog of the equivariant Riemann curvature $R_{\mathfrak{g}}(\mathbf{X}) = (\nabla - \iota(\mathbf{X}))^2 + \mathcal{L}_{\mathfrak{g}}$. The \mathfrak{g} -equivariant eta invariant is now defined as the convergent power series

$$\eta_{\mathbf{X}}(D) := \int_0^\infty \frac{1}{\sqrt{\pi t}} \text{tr} \left(D_{\frac{\mathbf{X}}{t}} e^{-t\mathcal{H}_{\frac{\mathbf{X}}{t}}} \right) dt. \quad (7.29)$$

Note that this encodes the Clifford multiplication in the fiber dimensions.

Relating eta form to eta invariant. Let $Y^{11} \rightarrow X^{11-d}$ be a totally geodesic fiber bundle with compact, oriented, odd-dimensional fiber M^d and compact holonomy group G . Let \overline{D}_M be a family of Dirac operators induced by a G -equivariant Dirac operator D_M on a G -equivariant Clifford module E_M . Let V be an element of \mathfrak{g} , the Lie algebra of the isometry group of the fiber. Then the η -form $\widehat{\eta}(\overline{D}_M)$ of the family \overline{D}_M is related to the g -equivariant η -invariant $\eta_V(D_M)$ of D_M by ([81])

$$\widehat{\eta}(\overline{D}_M) = \frac{1}{2} \eta_V(D_M) . \quad (7.30)$$

Note that by definition, $\eta_V(D_M) \in \mathbb{C}[\mathfrak{g}^*]$ is Ad_G -invariant. As the image of $\eta_V(D_M)$ under the Chern-Weil homomorphism, $\widehat{\eta}(\overline{D}_M)$ is thus a characteristic form of the bundle $P \rightarrow X^{11-d}$. In particular, it is closed and $[\widehat{\eta}(\overline{D}_M)] \in H^*(X^{11-d}; \mathbb{R})$ is independent of the connection on P . However, a change of the metric on M^d will in general affect $[\widehat{\eta}(\overline{D}_M)]$.

Eta form on the circle. This is considered in [81]. Let $D = -i\left(\frac{\partial}{\partial\theta} + c\right)$ be the Dirac operator on the trivial bundle $\mathcal{E} := S^1 \times \mathbb{C}$ over $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with respect to the connection $\nabla_{\frac{\partial}{\partial\theta}} = \frac{\partial}{\partial\theta} + ic$. Define an \mathbb{R} -operation on \mathcal{E} by

$$g(\theta, z) := (\theta + g, e^{i\lambda g} z), \quad \text{for } g \in \mathbb{R} . \quad (7.31)$$

The Dirac operator with respect to the trivial Spin structure corresponds to $c = \lambda = 0$ and is $\mathbb{R}/2\pi\mathbb{Z}$ -equivariant. The Dirac operator with respect to the non-trivial Spin structure corresponds to $c = -\lambda = \frac{1}{2}$ and is $\mathbb{R}/4\pi\mathbb{Z}$ -equivariant. Then $D(e^{i\nu\theta}) = (\nu + c)e^{i\nu\theta}$ and $g^*(e^{i\nu\theta}) = e^{i\nu(\theta-g)+i\lambda g}$. For the untwisted Dirac operators D_1 and D_2 with respect to the trivial and nontrivial Spin structures on S^1 , respectively, we have

$$\eta_V(D_1) = i \frac{\frac{x}{2} - \tan \frac{x}{2}}{\frac{x}{2} \tan \frac{x}{2}} = 2 \sum_{k=1}^{\infty} \zeta(-k) \frac{(-ix)^k}{k!} , \quad (7.32)$$

and

$$\eta_V(D_2) = i \frac{\frac{x}{2} - \sin \frac{x}{2}}{\frac{x}{2} \sin \frac{x}{2}} = 2 \sum_{k=1}^{\infty} \zeta(-k, \frac{1}{2}) \frac{(-ix)^k}{k!} . \quad (7.33)$$

Eta forms over principal circle bundles. Let $Y^{11} \rightarrow X^{10}$ be an S^1 -principal bundle. If the associated Hermitian line bundle \mathcal{L} has curvature $F^{\mathcal{L}} \in i\mathcal{A}^2(X^{10})$, its first Chern form is $c_1(\mathcal{L}) = -\frac{1}{2\pi i}[F^{\mathcal{L}}]$ so that $\text{ch}(\mathcal{L}) = e^{2\pi i c_1(\mathcal{L})}$. Then the curvature of Y^{11} itself equals $\Omega = iF \frac{\partial}{\partial\theta}$. Let D_{S^1} be the equivariant Dirac operator on S^1 . Then D_{S^1} induces a family \overline{D}_{S^1} of fiber-wise Dirac operators on Y^{11} . Then the η -form of this family represents the characteristic class [81]

$$\begin{aligned} [\widehat{\eta}(\overline{D}_{S^1})] &= \frac{1}{2} [\eta(D_{S^1})] \\ &= \sum_{k=0}^{\infty} \frac{B_{k+1}(\lambda - [-c]) + B_{k+1}(1 + \lambda + [c]) - 2(c + \lambda)^{k+1}}{(k+1)!} (2\pi i c_1(\mathcal{L}))^k . \end{aligned} \quad (7.34)$$

Here $B_k(r)$ are Bernoulli polynomials, defined by $\frac{te^{rt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(r) \frac{t^k}{k!}$, which coincide with Bernoulli numbers when $r = 0$, $B_k = B_k(0)$, i.e. $B_0(r) = 1$, $B_1(r) = r - \frac{1}{2}$, $B_2(r) = r^2 - r + \frac{1}{6}$. For example, for the operator induced by D_1 in (7.32),

$$[\widehat{\eta}(\overline{D}_1)] = \frac{1}{2} \left[\frac{\frac{F^{\mathcal{L}}}{2} - \tanh \frac{F^{\mathcal{L}}}{2}}{\frac{F^{\mathcal{L}}}{2} \tanh \frac{F^{\mathcal{L}}}{2}} \right] = - \sum_{k=0}^{\infty} \zeta(-k) \frac{(2\pi i c_1(\mathcal{L}))^k}{k!} . \quad (7.35)$$

This was also derived in [168] [169].

Example 1: Heisenberg manifolds. The 11-dimensional Heisenberg manifold can be described as an S^1 -bundle Y^{11} over the ten-torus T^{10} , with Chern class $c_1(Y^{11}) = -(d\theta_1 \wedge d\theta_2 + \cdots + d\theta_9 \wedge d\theta_{10})$, where $\theta \in \mathbb{R}/\mathbb{Z}$ parametrizes the i th factor of $T^{10} = \underbrace{S^1 \times \cdots \times S^1}_{10}$. Since the torus is flat, it has $\hat{A}(T^{10}) = 1$, so

that the only term contributing to the adiabatic limit is $c_1(Y)^5 = (-1)^5 5! d\theta_1 \wedge \cdots \wedge d\theta_{10}$. For the untwisted Dirac operator $D_{Y,t}$ determined by $P_{\text{Spin}} Y^{11}$ on the bundle Y^{11} with fiber of length t , the adiabatic limit is

$$\lim_{t \rightarrow 0} \eta(D_{Y,t}) = \begin{cases} 2\zeta(-5) & \text{if the Spin structure restricted to the fiber is trivial, and} \\ 2\zeta(-5, \frac{1}{2}) & \text{otherwise.} \end{cases} \quad (7.36)$$

Now consider the seven-dimensional case, i.e. Y^7 is a circle bundle over the six-torus T^6 . In this case, the adiabatic limit vanishes.

Example 2: S^3 -bundles. Let D be the untwisted Dirac operator on S^3 . The equivariant eta invariant corresponding to

$$V = \begin{pmatrix} 0 & x & & \\ -x & 0 & & \\ & & 0 & y \\ & & -y & 0 \end{pmatrix} \in \mathfrak{so}(4) \cong \mathfrak{spin}(4) \quad (7.37)$$

is [81]

$$\eta_V = 2 \sum_{k,l=1}^{\infty} (-1)^{k+l} \left(\frac{B_{2k+2l}(1/2)}{(2k+2l)!} - \frac{B_{2k}(1/2)B_{2l}(1/2)}{(2k)!(2l)!} \right) x^{2k-1} y^{2l-1}. \quad (7.38)$$

Now we would like to consider S^3 bundles $Y^{11} \rightarrow X^8$. The first Pontrjagin class $p_1(E_4)$ and the Euler class $\chi(E_4)$ of the $\text{Spin}(4)$ bundle correspond to the polynomials

$$V \mapsto (-2\pi i)^{-2}(x^2 + y^2), \quad V \mapsto (-2\pi i)^2 xy. \quad (7.39)$$

The η -form for the induced Dirac operator \overline{D} on S^3 -bundle is (without the factors of $2\pi i$)

$$\begin{aligned} \hat{\eta}(\overline{D}) &= \hat{\eta}_4 + \hat{\eta}_8 + \cdots \\ &= \frac{\chi(E_4)}{2^7 \cdot 3 \cdot 5} - \frac{\chi(E_4)p_1(E_4)}{2^9 \cdot 3 \cdot 5 \cdot 7}. \end{aligned} \quad (7.40)$$

Now X^8 is eight-dimensional so that the possible components of the eta-forms appearing in (7.22) are 0, 4, and 8,

$$\begin{aligned} \lim_{t \rightarrow 0} \eta(D_{Y,t}) &= \int_{X^8} \hat{A}_8(X^8) + \hat{A}_4(X^8) \wedge \hat{\eta}_4 + \hat{\eta}_8 \\ &= \int_{X^8} \frac{(7p_1(X^8)^2 - 4p_2(X^8))}{2^7 \cdot 3^2 \cdot 5} - \frac{\chi(E_4)p_1(X^8)}{2^{14} \cdot 3^3 \cdot 5} - \frac{\chi(E_4)p_1(E_4)}{2^9 \cdot 3 \cdot 5 \cdot 7}. \end{aligned} \quad (7.41)$$

Spherical space forms. Consider the three-dimensional Spin spherical space form $M^3 = S^3/\Gamma$. The value of the equivariant eta invariant depends on the Spin structure. For example, consider $\mathbb{R}P^3 = S^3/\Gamma$, where $\Gamma = \{\text{id}, -\text{id}\} \subset SO(4)$. The finite group Γ acts freely on S^3 preserving the orientation. There are two lifts of Γ to $\text{Spin}(4)$

$$\tilde{\Gamma}_1 := \{1, e_1 e_2 e_3 e_4\}, \quad \tilde{\Gamma}_2 := \{1, -e_1 e_2 e_3 e_4\}, \quad (7.42)$$

both in $\text{Spin}(4) \subset C\ell(\mathbb{R}^4)$. When lifted to the Spin bundle, these give two different Spin structures on the quotient (see section 2.2.1). Let D be the Dirac operator on S^3 and let D_1 and D_2 be the untwisted Dirac

operators on $\mathbb{R}P^3$. The equivariant eta invariants are given by (see [81])

$$\eta_V(D_1) = \frac{1}{2}\eta_V(D) - \frac{1}{4\cos\frac{x}{2}\cos\frac{y}{2}}, \quad (7.43)$$

$$\eta_V(D_2) = \frac{1}{2}\eta_V(D) + \frac{1}{4\cos\frac{x}{2}\cos\frac{y}{2}}, \quad (7.44)$$

where x and y are the entries in the matrix (7.37).

Effect of parity on Dirac structures. In M-theory, the action and equations of motion admit a parity symmetry given by an odd number of space or time reflections together with $C_3 \mapsto -C_3$ [60]. Thus, in the Riemannian case, this is orientation reversal. In the presence of the one-loop term and of E_8 bundles, this parity symmetry takes the form [165] [54] $G_4 \mapsto -G_4$ and $a \mapsto \lambda - a$, where a is the characteristic class of the E_8 bundle and λ is the first Spin characteristic class. This parity is also a reflection on the E_8 class a if M-theory is taken on String eleven-dimensional manifolds such as in [149].

We saw at the end of section 2.3 how the Spin structures in ten and eleven dimensions get modified when the orientation of the manifold is reversed. Let $-Y^{11}$ denote the eleven-dimensional manifold with the opposite orientation and Spin structure. Then the Dirac structures on Y^{11} and $-Y^{11}$ can be related as [39]

$$\mathcal{S}(-Y^{11}) \cong \mathcal{S}(Y^{11})^{\text{op}}, \quad (7.45)$$

where $\mathcal{S}(-Y^{11})$ is the Clifford module corresponding to the orientation-reversed manifold $-Y^{11}$ and $\mathcal{S}(Y^{11})^{\text{op}}$ is obtained from $\mathcal{S}(Y^{11})$ by reversing the homomorphism $c : \mathcal{C}\ell(TY^{11}) \rightarrow \text{End}(V)$, with V the vector bundle used to give the module structure. For us, this is the E_8 bundle, or the tangent bundle TY^{11} or a flat bundle, capturing the fundamental group. The eta-forms satisfy $\hat{\eta}(\mathcal{E}^{\text{op}}) = -\hat{\eta}(\mathcal{E})$, where \mathcal{E} collectively corresponds to the geometric family as in [39]. When the eleven-dimensional manifold Y^{11} is a product or a bundle, we can apply the construction to the fiber (which could be a spherical space form) and the base. An analogous situation in type IIA string theory for the B -field is considered in [148].

7.4 The fields via eta forms, gerbes, and Deligne cohomology

The disk bundle and the circle bundle. Let $\mathbb{D}^2 \rightarrow Z^{12} \xrightarrow{\pi_D} X^{10}$ be the disk bundle corresponding to the circle bundle $S^1 = \partial\mathbb{D}^2 \rightarrow Y^{11} = \partial Z^{12} \xrightarrow{\pi_S} X^{10}$. Assume that X^{10} is compact and Spin and fix a Spin structure on $T\mathbb{D}^2$. Let $g_{\mathbb{D}^2}$ be a metric on $T\mathbb{D}^2$ and let g_{S^1} be the restriction to the circle. Assume that there is a neighborhood $[-1, 0] \times Y^{11}$ of Y^{11} in Z^{12} such that for any $x \in X^{10}$, on $\pi_D^{-1}(x) \cap ([0, 1] \times Y^{11}) = [-1, 0] \times S_x^1$, $g_{\mathbb{D}^2}$ takes the form $g_{\mathbb{D}^2}|_{[-1, 0] \times S_x^1} = dt^2 \oplus g_{S_x^1}$.

- Let $S\mathbb{D}^2$ be the spinor bundle of $(T\mathbb{D}^2, g_{\mathbb{D}^2})$. Since the disk is even-dimensional, the Spin bundle splits canonically into positive and negative Spin bundles $S\mathbb{D}^2 = S^+\mathbb{D}^2 \oplus S^-\mathbb{D}^2$.
- Let E be a complex vector bundle over Z^{12} . Let g_E be a metric on E such that $g_E|_{[-1, 0] \times S^1} = \pi^*g_E|_{S^1}$.
- Let ∇^E be a Hermitian connection on E such that $\nabla^E|_{[-1, 0] \times S^1} = \pi^*\nabla^E|_{S^1}$.
- Let SS^1 be the Spin bundle of (TS^1, g_{S^1}) . Associated to $g_{S_y^1}$, $g_E|_{S_x^1}$, and $\nabla^E|_{S_x^1}$ are canonically defined (twisted) Dirac operators $D_x^{S^1, E}$.

For any $x \in X^{10}$, associated to $g_{\mathbb{D}_x^2}$, $g_E|_{\mathbb{D}_x^2}$ and $\nabla^E|_{\mathbb{D}_x^2}$, there is also a twisted Dirac operator

$$D_x^{\mathbb{D}^2, E} : \Gamma(S^+\mathbb{D}_x^2 \otimes E|_{\mathbb{D}_x^2}) \rightarrow \Gamma(S^-\mathbb{D}_x^2 \otimes E|_{\mathbb{D}_x^2}). \quad (7.46)$$

On $U_x = [-1, 0] \times S_x^1$, this takes the form $D_x^{\mathbb{D}^2, E} = c(\partial_t) \left(\partial_t + D_x^{S^1, E} \right)$.

Fields in Deligne cohomology from eta forms. Eta-forms behave nicely under inclusion of open subsets in a space. If $\mathcal{E}|_U$ is the restriction of the geometric family \mathcal{E} of Dirac operators to the subset $U \subseteq X$, then (see [39])

$$\widehat{\eta}(\mathcal{E}|_U) = \widehat{\eta}(\mathcal{E})|_U. \quad (7.47)$$

Then there is a corresponding Dirac operator D_α on each open subset U_α . This allows for a description in terms of Deligne cohomology [113] [39]. Thus, we will refine the discussion of section 7.3.

The degree zero component $\widehat{\eta}_0$ is half of the Atiyah-Patodi-Singer eta invariant of D_α . Consider an S^3 bundle $Y^{11} \rightarrow X^8$. If $U_\alpha \cap U_\beta \neq \emptyset$ then $\widehat{\eta}_\beta^0|_{U_\alpha \cap U_\beta} - \widehat{\eta}_\alpha^0|_{U_\alpha \cap U_\beta}$ is an integer-valued function on $U_\alpha \cap U_\beta$. Define $f_\alpha : U_\alpha \rightarrow S^1$ by $f_\alpha = e^{2\pi i \widehat{\eta}_\alpha^0}$. Then if $U_\alpha \cap U_\beta \neq \emptyset$ then $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$, so that the functions $\{f_\alpha\}_{\alpha \in I}$ glue together to form a function $f : X^8 \rightarrow S^1$ such that $f|_{U_\alpha} = f_\alpha$. Let $[S^1] \in H^1(S^1; \mathbb{Z})$ be the fundamental class of S^1 . Then, applying [113], $f^*[S^1] \in H^1(X^8; \mathbb{Z})$ is represented in real cohomology by the closed form

$$\frac{1}{2\pi i} \ln f = \left(\int_{S^3} \widehat{A}(R_{S^3}) \wedge \text{ch}(F^V) \right)_{[1]} \in \Omega^1(X^8). \quad (7.48)$$

Next we consider the gerbe on X^{10} via the circle bundle $\pi : Y^{11} \rightarrow X^{10}$. The ‘phase’ $\frac{D}{|D|}$ of the Dirac operator D has eigenvalues ± 1 . Thus on $U_\alpha \cap U_\beta$ the operator $\frac{D_\beta}{|D_\beta|} - \frac{D_\alpha}{|D_\alpha|}$ has eigenvalues 0, 2 and -2 . Let P_0 , P_2 and P_{-2} , respectively, be the images of the projections of the corresponding eigenspaces. These are finite-dimensional vector bundles on $U_\alpha \cap U_\beta$. From [113], this gives a gerbe with connection on X^{10}

$$d\widehat{\eta}^2 = \left(\int_{S^1} \widehat{A}(R_{S^1}) \wedge \text{ch}(F^V) \right)_{[3]} \in \Omega^3(X^{10}) \quad (7.49)$$

with data:

1. Line bundle $L_{\alpha\beta} = \Lambda^{\max}(P_2) \otimes (\Lambda^{\max}(P_{-2}))^{-1}$ with a unitary connection $\nabla_{\alpha\beta}$, inherited from the projected connections on P_2 and P_{-2} .
2. The curvature of $\nabla_{\alpha\beta}$ is $F_{\alpha\beta} = \widehat{\eta}_\alpha^2 - \widehat{\eta}_\beta^2$
3. a nowhere zero section $\theta_{\alpha\beta\gamma}$ on $L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha}$ if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.
4. F_α , the 2-form component of the eta-form.

As a rational cohomology class, (7.49) lies in the image of $H^3(X^{10}; \mathbb{Z}) \rightarrow H^3(X^{10}; \mathbb{Q})$. This is the H -field on X^{10} .

Next we consider examples where the dimension of the fiber is even. Here we can start in twelve or in eleven dimensions. In the first case we can a flat B -field in type IIA string theory. Let $\pi_D : Z^{12} \rightarrow X^{10}$ be a disk bundle with fiber the disk \mathbb{D}^2 . Let $D^\mathbb{D}$ denote the family $D^\mathbb{D} = \{D_x\}_{x \in X^{10}}$ of Dirac operators, with D_x acting on $C^\infty(\mathbb{D}_x^2; E|_{\mathbb{D}_x^2})$. Then

$$d\widehat{\eta}_1 = \left(\int_{\mathbb{D}^2} \widehat{A}(R_{\mathbb{D}^2}) \wedge \text{ch}(F^V) \right)_{[2]} \in \Omega^2(X^{10}). \quad (7.50)$$

Next we consider M-theory on the two-torus, i.e. take Y^{11} to be the total space of a torus bundle $\pi' : Y^{11} \rightarrow X^9$, with fiber T^2 . Let D' be the family of Dirac operators $D' = \{D'_x\}_{x \in X^9}$ of Dirac operators, with D'_x acting on $C^\infty(T_x^2; E|_{T_x^2})$. Then we similarly get a flat B -field, but now on X^9 .

New contributions occur from the kernel of the Dirac operator. Assume that $\ker(D^D)$ has a constant rank, i.e. is a \mathbb{Z}_2 -graded vector bundle on X^{10} . The connection $\nabla^{\pi_* E}$ projects into a connection $\nabla^{\ker(D^D)}$ on $\text{Ker}(D^D)$ with curvature $F^{\ker(D^D)}$. Then

$$d\widehat{\eta} = \int_{\mathbb{D}^2} \widehat{A}(R_{\mathbb{D}^2}) \wedge \text{ch}(F^V) - \text{ch}(F^{\ker(D^D)}). \quad (7.51)$$

Assume further that for each $x \in X^{10}$, the index of D_x^D vanishes in \mathbb{Z} , so that the bundles $\ker(D^D)_+$ and $\ker(D^D)_-$ have the same rank. The corresponding connections $\nabla^{\ker(D^D)_{\pm}}$ have curvatures F^{\pm} . We choose an open covering of X^{10} by open subsets such that there is an isometric isomorphism $\kappa_{\alpha} : \ker(D^D)_+|_{U_{\alpha}} \rightarrow \ker(D^D)_-|_{U_{\alpha}}$. There is a 1-form $CS_1^{\alpha} \in \Omega^1(U_{\alpha})$ such that

$$dCS_1^{\alpha} = \text{ch}(F^+) - \text{ch}(\kappa_{\alpha}^{-1}F^-\kappa_{\alpha}) = \text{ch}(F^{\ker(D^D)}) . \quad (7.52)$$

Then this ‘Chern-Simons form’ combines with the eta-form to give

$$d(\widehat{\eta}_1 + CS_1^{\alpha}) = \left(\int_{\mathbb{D}^2} \widehat{A}(R_{\mathbb{D}^2}) \wedge \text{ch}(F^V) \right) \in \Omega^2(X^{10}) . \quad (7.53)$$

This gives an extra contribution to the flat B -field, given by the above Chern-Simons form coming from the kernel of the Dirac operator. The ‘descent relations’ can be found in [113].

Similarly we can consider M-theory on a two-torus. In this case, we have

$$d(\widehat{\eta}_1 + CS_1^{\alpha}) = \left(\int_{T^2} \widehat{A}(R_{T^2}) \wedge \text{ch}(F^V) \right) \in \Omega^2(X^9) . \quad (7.54)$$

Filtration in K-theory. The above gerbes have obstructions which can be seen using the filtration in K-theory that leads to the Atiyah-Hirzebruch Spectral Sequence [113] [39]. For a positive integer p and an element in K-theory $\psi \in K^*(X^{10})$, we say $\psi \in K_p^*(X^{10})$ if $f^*\psi = 0$ for all branes M of dimension $< p$ and continuous maps $f : M \rightarrow X^{10}$. Then there is a natural decreasing filtration

$$\cdots \subseteq K_p^*(X^{10}) \subseteq K_{p-1}^*(X^{10}) \subseteq \cdots \subseteq K_0^*(X^{10}) = K^*(X^{10}) \quad (7.55)$$

which preserves periodicity and the ring structure. Consider a class $x \in K_p^r(X)$. A class in $E_2^{p,r-p}$ is $z \in H^p(X; \mathbb{Z})$. Its image $z_{\mathbb{Q}} \in H^p(X; \mathbb{Q})$ of z in rational cohomology satisfies $\text{ch}_p(x) = z_{\mathbb{Q}}$.

Example: circle bundle. Using the calculation of the eta-form for circle bundles (7.34), the Deligne cohomology class corresponding to z_{2m} is [39]

$$[\widehat{\eta}(z_{2m})] = a \left[\frac{B_{m+1}}{(m+1)!} c_1(\omega)^m \right] \in H_{Del}^{2m+1}(X^{10}) . \quad (7.56)$$

In particular, the curvature of the Deligne class is zero. Consider the case $m = 1$. The Deligne cohomology class of z_2 corresponds to a gerbe of the geometric family. The index gerbe is then [39]

$$[\widehat{\eta}(z_2)] = a \left[\frac{1}{12} c_1(\omega) \right] . \quad (7.57)$$

For example, take $X = \mathbb{CP}^1$ and let $Y \rightarrow X$ be the square of the Hopf bundle. Then $c_1(Y) = 2$ and $[\widehat{\eta}(z_2)] \cong [1/6]_{\mathbb{R}/\mathbb{Z}}$ under the isomorphism $H_{Del}^3(\mathbb{CP}^1) \cong \mathbb{R}/\mathbb{Z}$.

Kaluza-Klein modes and Adams operations. In [54], the massive modes of the spin 1/2 fermions were used in the expressions for the index of the Dirac operator. Their idea could be summarized as follows. The M-theory circle bundle can be viewed as unit vectors in a complex line bundle \mathcal{L} . So functions on Y^{11} that transform as $e^{-ik\theta}$ under rotations of the circle can be viewed as sections of \mathcal{L}^k over X . The space of functions on Y^{11} can be written in terms of the space of sections of \mathcal{L}^k on X^{10} as $\text{Func}(Y^{11}) = \bigoplus_{k \in \mathbb{Z}} \Gamma(X^{10}, \mathcal{L}^k)$. Here $\text{Func}(Y^{11})$ is the space of functions on Y^{11} , and $\Gamma(X^{10}, \mathcal{L}^k)$ the space of sections of \mathcal{L}^k .

One can think of this line bundle as generating the higher modes. One can get from level $n = 1$ all the other levels. Mathematically, one could think of using an Adams operation on the line bundle,

$$\psi^k(\mathcal{L}) = \mathcal{L}^k \quad (7.58)$$

so the bundles on X^{10} are tensored with \mathcal{L}^m . This is a cohomology operation on $K(X^{10})$, i.e. $\psi^k : K^0(X^{10}) \rightarrow K^0(X^{10})$, $k = 0, 1, 2, \dots$. The Adams operations satisfy $\psi^k \circ \psi^l = \psi^{kl} = \psi^l \circ \psi^k$. This property is satisfied in our context of the M-theory line bundle and corresponds to additivity and commutativity of the Fourier modes. The coupling of this line bundle to the vector bundles already in X^{10} might give an idea about the extension beyond K-theory. We will consider this elsewhere.

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